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# Allocation of Fixed Costs in the Presence of Production Trade-offs in Data Envelopment Analysis

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## Abstract

In this paper, we present a new case for the topic of fixed cost allocation in the presence of production trade-offs in Data Envelopment Analysis (DEA). To this end, we use the principle of unchanged efficiency and propose a fixed cost allocation model in such a way that the efficiency scores of Decision Making Units (DMUs) do not change before and after the allocation of fixed costs. By considering production trade-offs on the input and output components, we incorporate the importance of these inputs and outputs in the fixed cost allocation model. By treating fixed costs as a new input, we incorporate the importance of this new input by defining production trade-offs. According to the proposed fixed cost allocation plan, costs are allocated among efficient and inefficient units. An application of approximation for the data set is employed in the petrochemical industry, and the results of the models are presented.

**Keywords:** Data envelopment analysis, Fixed cost allocation, Production trade-offs.

## 1 | Introduction

Fixed cost allocation is a critical issue for many managers. It occurs in the construction of a common platform within an organization. All sections related to the platform must share the fixed cost of the platform. Several approaches have been proposed to address fixed cost allocation. One of these approaches is Data Envelopment Analysis (DEA). A key issue in fixed cost allocation plans is fair allocation, which can lead to the organization's growth and survival. This allocation should be based on the potential and capacity of each company. These companies operate under a unified management. Typically, providing a fair fixed cost

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allocation plan is a challenging yet important task. Such a plan should help improve organizational performance and prevent resource wastage. Research shows that fixed costs can also influence product pricing strategies, especially when companies face liquidity constraints [1], [2]. Therefore, fair cost allocation among companies operating in the same field under unified management is important [3]. Inappropriate fixed cost allocation may lead to consequences such as resource waste. For example, when the central bank management attempts to share operational fees with its branches, a conventional cost allocation mechanism might require smaller banks to pay higher fees compared to larger banks, leading smaller banks toward closure. This outcome is not favorable for senior bank managers, so the fixed cost allocation plan is conducted with a strategy to improve the efficiency of all banks, ensuring that the allocated costs enhance bank performance. One suitable technique for providing a fixed cost allocation plan is DEA. One strategy for cost allocation is to ensure that efficiency does not worsen after allocation. Cook and Zhu [4] proposed a model for fair cost allocation based on the principle of unchanged efficiency. Lin [5] showed that the Cook and Zhu [4] model does not have a solution when adding certain specific constraints. Mostafaei [6] proposed a model based on unchanged efficiency, aiming to change the class interval to unit scale. Jahanshahloo et al. [7] presented a model based on the sum of common weights in DEA for providing a fixed cost allocation plan, considering the strategy of unchanged efficiency. Li et al. [8] proposed a model for a fixed cost allocation plan for two-stage networks, where all units became efficient after the allocation plan. Another strategy for fixed cost allocation is based on the decision-maker's preference information, where some units' efficiency may worsen while others improve, with the total cost allocated among units. This method is conducted interactively in DEA, based on preferred goals [9], [10]. Li .et al. [11] provided a degree of satisfaction approximation to obtain a unique allocation plan based on the principle of effectiveness. Chu.et.al [12] proposed a competition between the two stages of a two-stage network based on a leader-follower model to obtain a unique cost allocation plan, following the principle of effectiveness. Xu .et.al [13] presented a unique cost allocation plan based on DEA from the perspective of inequality aversion. They demonstrated that their plan is unique, allocating costs to units under both optimistic and pessimistic scenarios. The models proposed by Cook and Kress [14] and Cook and Zhu [4] were based on the principle of unchanged efficiency. The principle of effectiveness implies that the efficiency of all units should not decrease after the allocation of fixed costs compared to before the allocation; in fact, the term "effective" refers to maintaining or enhancing efficiency. This approach may lead to a non-unique cost allocation plan, and additional criteria need to be introduced to achieve a unique cost allocation plan. Li.et.al [11] proposed a satisfaction approximation to obtain a unique cost allocation plan based on the principle of effectiveness. It should be noted that using different methods to propose a unique cost allocation plan might result in unfair allocation schemes, posing risks to some units. The cost allocation plan must be equitable and acceptable to all Decision-Making Units (DMUs).

One way to incorporate preference information in evaluating the efficiency of DMUs is through weight restrictions. In this approach, the relative importance of input and output components is determined by imposing weight restrictions. By applying weight constraints, we can take into account the importance of input and output components relative to each other in evaluating the performance of units. Weight restrictions are applied in multiplicative models. However, the equivalent problem of weight restrictions in multiplicative models is production trade-offs in envelopment models. By placing production trade-offs on the input and output components, we can account for their importance and relationships in envelopment models. The concept of trade-off refers to the compromises or exchanges between different inputs and outputs that occur during the efficiency evaluation process. These interactions play a crucial role in decision-making and improving unit performance. In defining trade-off, an improvement in one output or a reduction in one input might lead to a decrease in other outputs or an increase in other inputs. In other words, improvement in one aspect might come at the expense of a decrease in others. A trade-off in an input might result in an increase in other inputs; for instance, reducing labor costs might lead to an increase in capital expenses. Similarly, a trade-off in outputs could mean an improvement in one output accompanied by a reduction in others, such as increasing the production of one product might lead to a decrease in the quality of another product. Trade-offs between inputs and outputs suggest that a reduction in one input might result in a reduction in an output, or increasing an output might require an increase in an input. In DEA, trade-offs facilitate informed decision-making and enhance efficiency by optimally allocating resources and managing limited resources. Podinovski [15] examined how interactions between inputs and outputs can be modeled using relative weights in DEA models. He demonstrated that by imposing appropriate constraints, the balances between variables could be promptly integrated into the model. For example, if a reduction in one input results in an increase in another input, this relationship can be represented by imposing linear constraints on the related weights in the DEA model. Podinovski [16] explored nonlinear balances between variables and proposed methods for modeling these balances. Podinovski [17] analyzed the directions of trade-offs and showed how these directions can be identified in DEA models. He also examined optimal weights in DEA models with weight constraints, stating that optimal targets are achieved across all units under weight constraints. Podinovski [16] provided methods for calculating efficiency and obtaining efficient targets in DEA models in the presence of weight constraints. They presented a two-stage model for finding feasible and efficient targets for all units, incorporating production technology with trade-offs. Podinovski and Bouzdine [18] explored the issue of unrestricted and free production in envelopment models with trade-offs, showing that with unlimited outputs, multiplicative models might become infeasible. In 2015, they investigated consistent weight constraints in DEA, examining the necessary and sufficient conditions for model feasibility in the presence of trade-offs between inputs and outputs. They demonstrated how to solve models and compute efficiency when free or unlimited outputs are present. In 2016, Podinovski and Bouzdine [19] proposed a single-stage DEA model in the presence of trade-offs to obtain feasible targets in DEA. Atici and Podinovski [20] assessed the technical efficiency of decision-making units or various specialists, applying their model in the agricultural industry using the production trade-offs method. Podinovski et al. [21] addressed DEA models in the presence of production trade-offs when inputs and outputs are ratio data, evaluating the performance of secondary schools in England.

In this paper, we aim to present a fair fixed cost allocation plan based on the strategy of unchanged efficiency in the presence of production trade-offs. To achieve this, we first introduce the concept of production trade-offs on inputs and outputs, considering fixed costs as a new input in the model. In this context, we propose an algorithm to determine the allocated cost to units, taking into account all efficient and inefficient units.

The structure of this paper is as follows: in the second section, we introduce the topic of production trade-offs in DEA to consider the relative importance of inputs and outputs, demonstrating that taking production trade-offs into account is equivalent to considering weight restrictions in multiplicative models. In the third section, we present a fixed cost allocation plan in the presence of production trade-offs. In the fourth section, we apply the proposed approach to a dataset of refineries in Iran and provide a cost allocation plan for these refineries, presenting the results of our research.

# 2 | Production Trade-offs in DEA

Consider n DMUs represented as $DMU_j = (x_j, y_j)$. Each decision-making unit consumes an input vector $x_j \in R_+^m$ to produce an $y_j \in R_+^s$. The input and output vectors are non-negative, and at least one input and one output are strictly positive for at least one DMU. Suppose we have L assigned production trade-

offs denoted by $(\tau_t, \Gamma_t)$, where $t = 1, \dots, L$. The vectors $\tau_t \epsilon R^m$ and $\Gamma_t \epsilon R^s$ adjust the inputs and outputs, respectively. By assuming the feasibility of production trade-offs as described below, we can present the CCR model for evaluating the performance of the unit under assessment, $DMU_o = (x_o, y_o)$.

Podinovski [15] defined this principle as follows: for defining the feasibility of production trade-offs, assume $(X, Y) \in T$, the production possibility set, and that the production trade-off system $t = 1, \dots, L$ is expressed as $(\tau_t, \Gamma_t)$. For each $\alpha_t \geq 0$, the condition $(X + \alpha_t \tau_t, Y + \alpha_t \Gamma_t) \in T$ holds provided that

$$X + \alpha_t \tau_t \geq 0 \text{ and } Y + \alpha_t \Gamma_t \geq 0 . \quad (1)$$

Podinovski [22] presented *Model (2)* in an output-oriented nature for evaluating the efficiency of $DMU_o$ in the presence of production trade-offs under constant returns to scale, as follows.

$$\begin{aligned} &\text{Max } \varphi_o, \\ &\text{s.t. } \sum_{j=1}^{n} \lambda_{jo} x_j + \sum_{t=1}^{L} \alpha_{to} \tau_t \le X_o , \\ &\sum_{j=1}^{n} \lambda_{jo} Y_j + \sum_{t=1}^{L} \alpha_{to} \Gamma_t \ge \varphi_o Y_o , \\ &\lambda_{jo} \ge 0,\ j = 1, \dots, n,\ \alpha_{to} \ge 0, t = 1, \dots, L, \varphi_o \text{ free sign.} \end{aligned} \tag{2}$$

Podinovski [16] proposed the following to identify efficient targets in the presence of production trade-offs corresponding to $DMU_o$. Assume that the radial target $DMU_o$ obtained from *Model (2)* is $(X_o, \varphi_o^* Y_o) = (X_o^*, Y_o^*)$. To obtain non-negative targets, we first solve the following model.

$$\begin{aligned} &\text{Max } \sum_{r=1}^{s} \beta_r + \sum_{i=1}^{m} \gamma_i, \\ &\text{s.t. } \sum_{j=1}^{n} \lambda_{jo} x_{ij} + \sum_{t=1}^{L} \alpha_{to} \tau_{it} + \gamma_i + w_i = x_i^*, \\ &\sum_{j=1}^{n} \lambda_{jo} y_{rj} + \sum_{t=1}^{L} \alpha_{to} \Gamma_{rt} - \beta_r = y_r^* , \\ &\sum_{j=1}^{n} \lambda_{jo} x_{ij} + \sum_{t=1}^{L} \alpha_{to} \tau_{it} + w_i \ge 0, \\ &\lambda_{jo} \ge 0,\ j = 1, \dots, n,\ \gamma_i \ge 0, w_i \ge 0,\ i = 1, \dots, m,\ \beta_r \ge 0, r = 1, \dots s. \end{aligned} \tag{3}$$

Assume the optimal solution obtained from *Model (3)* is $(\bar{\lambda}, \bar{\beta}, \bar{\gamma}, \bar{w}, \bar{\alpha})$. We define the target corresponding to $Dmu_o$ obtained from *Models (2)* and *(3)* as follows.

$$\begin{aligned} &\bar{X} = \sum \bar{\lambda}_{jo} \bar{x_j} + \sum \bar{\alpha}_{to} \tau_t + \overline{w}, \\ &\bar{Y} = \sum \bar{\lambda}_{jo} \bar{y_j} + \sum \bar{\alpha}_{to} \Gamma_t, \\ &\text{Where } \bar{X} = X^* + \bar{\gamma} , \bar{Y} = Y^* + \bar{\beta}. \end{aligned} \tag{4}$$

The point $(\bar{X}, \bar{Y})$ is a Pareto efficient unit in the production technology in the presence of production trade-offs if $\varphi_o^* = 1$ and the vectors $\bar{\beta}, \bar{\gamma}$ are zero vectors; in this case, $DMU_o$ is efficient. Otherwise, it is inefficient. As we know, in traditional DEA models, the reference set for each $DMU_o$ is defined as follows.

$$M = \left\{ DMU_j \middle| \lambda_{jo}^* \ge 0, \text{ In one of the optimal solutions of the CCR model } \right\}. \tag{5}$$

This means if we set,, $(\bar{X}, \bar{Y}) = (\sum \bar{\lambda}_j X_j , \sum \bar{\lambda}_j Y_j)$, where $\lambda_k \ge 0$, then $DMU_k$ is included in the reference set associated with $DMU_o$. $DMU_k$ is efficient and is considered as the direction of efficiency for $DMU_o$. However, in the presence of production trade-offs, even if $\bar{\lambda}_k \ge 0$ in the optimal solution of *Model (3)*, $DMU_k$ might still be inefficient. The weight $\bar{\lambda}_o$ of $DMU_o$ in its own efficiency target may also be strictly positive, but this does not occur if the optimal solution from *Model (3)* is unique. Podinovski introduced *Model (5)* to find the maximum components $w_1$ based on the sum of solutions $\bar{\beta}, \bar{\gamma}$ from *Model (3)*, as outlined below.

$$\begin{aligned} &\text{Max } \sum_{i=1}^{m} w_i , \\ &\text{s.t. } \sum_{j=1}^{n} \lambda_{jo} x_{ij} + \sum_{t=1}^{L} \alpha_{to} \tau_{it} + w_i = x_{io}^* - \bar{\gamma}_i, \\ &\sum_{j=1}^{n} \lambda_{jo} Y_{rj} + \sum_{t=1}^{L} \alpha_{to} \Gamma_{rt} = y_r^* + \beta_r^* , \\ &\sum_{j=1}^{n} \lambda_{jo} x_{ij} + \sum_{t=1}^{L} \alpha_{to} \tau_{it} + w_i \ge 0, \\ &\lambda_{jo} \ge 0,\ j = 1, \dots, n,\ \alpha_{to} \ge 0, t = 1, \dots, L,, \\ &w_i \ge 0, i = 1, \dots, m. \end{aligned} \tag{6}$$

Note that $\bar{x}_i - \bar{\gamma}_i \ge 0$. Consider an optimal solution of *Model (6)* as $(\hat{\lambda}_o, \hat{\alpha}_o, \hat{w})$ .

**Theorem 1.** If $\lambda_k^o > 0$ in the optimal solution of *Model (6)*, then $DMU_k$ is a Pareto efficient unit in the presence of production trade-offs. According to *Theorem 1*, $DMU_k$ is included in the reference set of $DMU_o$. (For proof, refer to [16]).

# 3 | Allocation of Fixed Costs in the Presence of Production Trade-offs

Suppose we want to distribute a total fixed cost R among DMUs. Each $DMU_j$ is allocated a non-negative cost $r_j$ such that $\sum_{j=1}^{n} r_j = R$. The cost allocated to each $DMU_j$ is considered as a new input. The approximation provided in this paper, in the presence of production trade-offs, is based on the principle of maintaining efficiency unchanged. Each $DMU_j$ has no control over the allocated cost amount; therefore, their performance depends on their input and output levels. We assume that the efficiency of the unit under evaluation does not change after the allocation of the fixed cost $r_j$. *Model (2)*, in the presence of production trade-offs and cost $r_j$ corresponding to each $DMU_j$, for evaluating $DMU_o$, will be as follows.

$$\begin{aligned} &\text{Max } \varphi_o^{TCA},\\ &\text{s.t } \sum_{j=1}^{n} \lambda_{jo} x_{ij} + \sum_{t=1}^{L} \alpha_{to} \tau_{it} \le x_{io},\ i = 1, \dots, m,\\ &\sum_{j=1}^{n} \lambda_{jo} y_{rj} + \sum_{t=1}^{L} \alpha_{to} \Gamma_{rt} \ge y_{ro}\, \varphi_o^{TCA},\ r = 1, \dots, s,\\ &\sum_{j=1}^{n} \lambda_{jo} r_j + \sum_{t=1}^{L} \alpha_{to} \tau_{m+1t} \le r_o,\\ &\sum_{j=1}^{n} r_j = R,\\ &\lambda_{jo} \ge 0, r_j \ge 0,\ j = 1, \dots, n,\\ &\alpha_{to} \ge 0, t = 1, \dots, L. \end{aligned} \tag{7}$$

Assume $(\tilde{\varphi}_o^{TCA}, \tilde{\lambda}_j)$ is an optimal solution of *Model (7)* and $(\varphi_o^*, \lambda_j^*)$ is an optimal solution of *Model (2)*. Let the set NE represent the set of inefficient (non-extreme) units and E represent the set of efficient DMUs based on *Model (2)*. If the cost allocation satisfies $\sum \lambda_{jo}^* r_j \le r_o$ for all members $o \in NE$ , then $(\varphi_o^*, \lambda_j^*)$, is a solution for *Model (7)*, and we have $\tilde{\varphi}_o^{TCA} = \varphi_o^*$. If $DMU_o$ is inefficient, then $\lambda_j^* = 0$, and the equation $\sum \lambda_{jo}^* r_j \le r_o$ , can be written as $\sum_{j \in E} \lambda_{jo}^* r_j \le r_o$ for $o \in NE$. If $DMU_o$, is efficient before allocation, we have $\varphi_o^* = \tilde{\varphi}_o^{TCA} = 1$, and the cost allocation does not change the efficiency of the units. Therefore, if we make a cost allocation $(r_1, \dots, r_n)$ based on the principle of maintaining efficiency, the cost allocation must satisfy the inequality $\sum_{j \in E} \lambda_{jo}^* r_j \le r_o$ for all $o \in NE$, where E is the reference set corresponding to $DMU_o$, which was discussed in t he previous section. This reference set is obtained by solving *Model (6)*. Now, we provide a plan for fixed cost allocation in the presence of production trade-offs. Suppose we consider $\eta_j$ corresponding to the cost $r_j$ as a proportion of the total variable cost R allocated to $DMU_j$, which is determined as $\eta_j$ for $DMU_j$, and we have $\sum_{j=1}^{n} \eta_j = 1$. Considering the influence of the efficiency scores and input-output scales of the units in fixed cost allocation, we choose accordingly.

$$\eta_j = \frac{\left(\sum_{r=1}^{s} y_{rj} \cdot \sum_{i=1}^{m} x_{ij}\right)/\varphi_j^*}{\sum_{j=1}^{n}\left(\sum_{r=1}^{s} y_{rj} \cdot \sum_{i=1}^{m} x_{ij}\right)/\varphi_j^*}. \tag{8}$$

Therefore, a higher efficiency score $\varphi_j^*$ of $DMU_j$ will have less impact on the ratio $\eta_j$, and with larger input-output scales and better efficiency of $DMU_j$, the corresponding allocation ratio, $DMU_j$, will have a larger $\eta_j$ value. Thus, we define the ratio of the corresponding cost as follows.

$$\tilde{r}_j = \eta_j R = \frac{\left(\sum_{r=1}^{s} y_{rj} \cdot \sum_{i=1}^{m} x_{ij}\right)/\varphi_j^*}{\sum_{j=1}^{n}\left(\sum_{r=1}^{s} y_{rj} \sum_{i=1}^{m} x_{ij}\right)/\varphi_j^*} R. \tag{9}$$

To obtain the fixed cost allocation scheme for $DMU_j, j = 1, \dots, n$, we present the following model. If the allocated cost for each $DMU_j$ is considered as $\tilde{r}_j$, then all DMUs will pay a cost proportional to their relative efficiency and input-output scales, and we will have no changes such that these $\tilde{r}_j$ satisfy the principle of no change in efficiency. To address this, we define another principle known as the minimum deviation principle, which states that, without violating the principle of no change in efficiency, the difference between the allocated costs and the corresponding relative costs should be minimized as much as possible. The distance function can be expressed as follows.

$$D(r) = \sqrt{\sum_{j=1}^{n} (r_j - \tilde{r}_j)^2}.$$

We define it as follows, thus we can obtain the fixed cost allocation scheme by solving the following model.

$$\begin{aligned}
&\text{Min } \sum_{j=1}^{n}(r_j - \tilde{r}_j)^2, \\
&\text{s.t.} \quad \sum_{j \in EF} \lambda_{jo}^{*} r_j \le r_o \quad o \in NE, \\
&\sum_{j=1}^{n} r_j = R\,,\ r_j \ge 0, \quad j = 1, \dots, n.
\end{aligned} \tag{10}$$

In *Model (10)*, we modify the set EF based on *Model (4)*, and $\lambda_j^*$ and $\alpha_t^*$ are obtained from solving *Model (2)*. Therefore, we present the following algorithm for obtaining the fixed cost allocation scheme:

**Step 1.** Define the production trade-off matrices based on *Eq. (1)* and solve *Model (2)* to obtain the values of $\varphi_o^*$, $\lambda_{jo}^*$, $j = 1, \dots, n$, and $\alpha_{to}^*$, $t = 1, \dots, L$.

**Step 2.** Obtain the values of $\tilde{r}_j$ from *Eq. (9)*.

**Step 3.** Solve *Model (10)* to obtain the fixed cost allocation scheme.

It should be noted that *Model (7)* is a nonlinear model, requiring nonlinear algorithms for its solution. However, for solving *Model (10)*, we can replace the objective function with a linear one, such as $\min \sum_{j=1}^{n}|r_j - \tilde{r}_j|$ .

Therefore, we can obtain the fixed cost allocation scheme without solving *Model (7)*.

# 4 | Numerical Example

To illustrate the results of the models presented in this paper for the fixed cost allocation scheme, we use a numerical example from previous DEA studies. This example pertains to the work of Cook and Kress [14]. This dataset involves 12 DMUs, each consuming three inputs to produce two outputs. We assume that we want to allocate a fixed cost of 100 units among the DMUs. Initially, to incorporate the decision maker's perspective in the fixed cost allocation process, we use the method of production trade-offs. Accordingly, in line with the production trade-offs (1), we define the matrices ▯ and $\Gamma$ for inputs and outputs as follows:

Production trade-offs (1):

$$\tau = (0.5, 1, -2)\ , \Gamma = (2, 1).$$

The equivalent weight restriction of this production trade-off is as follows:

$$-2u_3 + u_2 + 0.5u_1 - v_1 - 2v_2 \le 0.$$

This production trade-off shows the relationship between the quantity of outputs produced from inputs and the corresponding input-output weights. Column seven of *Table 1* shows the results of *Model (2)* in evaluating the efficiency of the units in the presence of production trade-off (1). As observed, units 4, 5, 8, 9, and 12 are efficient, while the others are inefficient. Column eight of *Table 1* displays the reference set and the corresponding multipliers for the units in the reference set for each unit based on *Model (2)*. Additionally, the last column of the table shows the multipliers corresponding to only the considered production trade-offs. Now, to obtain the efficient targets for each unit in the presence of production trade-off (1), we consider *Model (3)*. These targets are frontier points from the production possibility set in the presence of the production trade-offs, which are efficient according to *Theorem 1*. Subsequently, to obtain the fixed cost allocation scheme in the presence of Production Trade-off (1), we first use *Eq. (8)* to adjust the $\eta_j$ values corresponding to each unit according to their efficiency scores from *Model (2)* and their input-output scales. These values are presented in the third column of *Table 3*. Also, the fourth column of *Table 3* shows the cost allocated to the units based on *Eq. (9)*. This cost is determined according to the efficiency values of the units in the presence of production trade-offs and the input-output sizes. Now, to obtain the fixed cost allocation scheme in the presence of production trade-offs, we solve *Model (10)*. The results obtained from *Model (10)* are in the last column of *Table 3*. This cost varies according to the efficiency scores, input-output sizes, and the production trade-off matrix. To conduct a sensitivity analysis of the model results relative to changes in the production trade-off matrix, we select these matrices differently. In the second choice, we select the production trade-off matrices as follows:

Production trade-offs (2):

$$\tau = (4, -3, 2), \;\; \Gamma = (1, -4).$$

The results related to this production trade-off are presented in *Tables 4-6*. Similar interpretations to those regarding the trade-off can be provided for this production trade-off. The corresponding weight restriction for this production trade-off is as follows.

$$2u_3 - 3u_2 + 4u_1 - v_1 + 4v_2 \leq 0,$$

As observed, in the second column of *Table 4*, units 4, 5, 8, 9, and 12 are efficient, and the other units are inefficient. The reference set corresponding to the inefficient units is listed in the third column of *Table 4*. *Table 5* shows the efficient targets corresponding to the decision-making units in the presence of production trade-off (2). *Table 4* presents the fixed cost allocation scheme. The second and third columns of *Table 4* show the values of $\eta_j$ and $\tilde{r}_j$ , corresponding to the efficiency scores in the presence of production trade-off (2) and the size of the inputs and outputs. The last column of *Table 6* displays the amount of fixed cost allocated to the units. This cost is provided under the assumption of no change in the efficiency of the units, and the efficiency scores of the units do not change after the fixed cost allocation, considering this cost as a new input.

**Table 1. Data set and the results of model (2) with trade-offs 1.**

| DMUs | Input 1 | Input 2 | Input 3 | Output 1 | Output 2 | The Efficiency Scores of Model (2) | Units in the Reference Set Based on the Model (2) | $\alpha^*_{to}$ |
|---|---|---|---|---|---|---|---|---|
| 1 | 350 | 39 | 9 | 67 | 751 | 0.6846 | $\lambda^*_9 = 0.8953, \lambda^*_{12} = 0.1047$ | 9.9159 |
| 2 | 298 | 26 | 8 | 73 | 611 | 0.8295 | $\lambda^*_4 = 0.8461, \lambda^*_9 = 0.1539$ | 8.6152 |
| 3 | 422 | 31 | 7 | 75 | 584 | 0.7697 | $\lambda^*_4 = 0.8031 \lambda^*_9 = 0.1969,$ | 13.2283 |
| 4 | 281 | 16 | 9 | 70 | 665 | 1 | $\lambda^*_4 = 1,$ | 0 |
| 5 | 301 | 16 | 6 | 75 | 445 | 1 | $\lambda^*_5 = 1$ | 0 |
| 6 | 360 | 29 | 17 | 83 | 1070 | 0.9978 | $\lambda^*_4 = 0.0141, \lambda^*_9 = 0.9859$ | 4.1269 |
| 7 | 540 | 18 | 10 | 72 | 457 | 0.924 | $\lambda^*_4 = 0.2162, \lambda^*_5 = 0.7838,$ | 2 |
| 8 | 276 | 33 | 5 | 78 | 590 | 1 | $\lambda^*_8 = 1$ | 0 |
| 9 | 323 | 25 | 5 | 75 | 1074 | 1 | $\lambda^*_9 = 1$ | 0 |
| 10 | 444 | 64 | 6 | 74 | 1072 | 0.8941 | $\lambda^*_{12} = 1$ | 0 |
| 11 | 323 | 25 | 5 | 25 | 350 | 0.331 | $\lambda^*_4 = 0.0412, \lambda^*_9 = 0.9588$ | 0.3711 |
| 12 | 444 | 64 | 6 | 104 | 1199 | 1 | $\lambda^*_5 = 1$ | 0 |

**Table 2. Targets of DMUs of model (3) with trade-offs 1.**

| DMUs | Input 1 | Input 2 | Input 3 | Output 1 | Output 2 |
|---|---|---|---|---|---|
| 1 | 340.6291 | 39 | 0 | 97.8687 | 1097.0059 |
| 2 | 298 | 26 | 8 | 87.9998 | 736.5459 |
| 3 | 295.882 | 31 | 0 | 97.4409 | 758.7415 |
| 4 | 281 | 16 | 9 | 70 | 665 |
| 5 | 301 | 16 | 6 | 75 | 445 |
| 6 | 324.4715 | 29 | 0 | 83.1832 | 1072.3631 |
| 7 | 540 | 18 | 10 | 77.9189 | 494.5686 |
| 8 | 276 | 33 | 5 | 78 | 590 |
| 9 | 323 | 25 | 5 | 75 | 1074 |
| 10 | 444 | 64 | 6 | 104 | 1199 |
| 11 | 323 | 25 | 5 | 75.5361 | 1057.5052 |
| 12 | 444 | 64 | 6 | 104 | 1199 |

**Table 3. The results of fixed cost allocation with trade-offs 1.**

| DMUs | Scale of Data | $\eta_j$ | $\tilde{r}_j$ | Fixed Cost Allocation |
|---|---|---|---|---|
| 1 | 475553.6079 | 0.10174021 | 10.174021 | 11.2757 |
| 2 | 273764.9186 | 0.058569423 | 5.8569423 | 9.3648 |
| 3 | 393841.7565 | 0.08425873 | 8.425873 | 9.5290 |
| 4 | 224910 | 0.048117374 | 4.8117374 | 8.7774 |
| 5 | 167960 | 0.035933458 | 3.5933458 | 5.3607 |
| 6 | 469150.1303 | 0.100370247 | 10.0370247 | 12.5405 |
| 7 | 325186.1472 | 0.06957051 | 6.957051 | 6.0994 |
| 8 | 209752 | 0.044874463 | 4.4874463 | 6.0105 |
| 9 | 405597 | 0.086773654 | 8.6773654 | 12.5944 |
| 10 | 658812.2134 | 0.140946661 | 14.0946661 | 6.0105 |
| 11 | 399924.4713 | 0.08556007 | 8.556007 | 12.4371 |
| 12 | 669742 | 0.143284986 | 14.3284986 | 0.0000 |
| sum | 4674195 | - | - | - |

**Table 4. Data set and the results of model (2) with trade-offs 2.**

| DMUs | The Efficiency Scores of Model (2) | Units in the Reference Set Based on the Model (2) | $\alpha_{to}^*$ |
|---|---|---|---|
| 1 | 0.7798 | $\lambda_8^* = 0.3106, \lambda_9^* = 0.3602, \lambda_{12}^* = 0.3292$ | 0.4411 |
| 2 | 0.9208 | $\lambda_5^* = 0.1377, \lambda_8^* = 0.6650, \lambda_9^* = 0.1673, \lambda_{12}^* = 0.0299$ | 1.4162 |
| 3 | 0.8775 | $\lambda_5^* = 0.6563, \lambda_{12}^* = 0.3438$ | 0.5 |
| 4 | 1 | $\lambda_4^* = 1,$ | 0 |
| 5 | 1 | $\lambda_5^* = 1$ | 0 |
| 6 | 0.9849 | $\lambda_8^* = 0.0226, \lambda_9^* = 0.7280, \lambda_{12}^* = 0.2494$ | 1.9695 |
| 7 | 0.8798 | $\lambda_5^* = 0.8333, \lambda_{12}^* = 0.1667$ | 2 |
| 8 | 1 | $\lambda_8^* = 1$ | 0 |
| 9 | 1 | $\lambda_9^* = 1$ | 0 |
| 10 | 0.8941 | $\lambda_{12}^* = 1$ | 0 |
| 11 | 0.3333 | $\lambda_9^* = 1$ | 0 |
| 12 | 1 | $\lambda_5^* = 1$ | 0 |

**Table 5. Targets of DMUs of model (3) with trade-offs 2.**

| DMUs | Input 1 | Input 2 | Input 3 | Output 1 | Output 2 |
|---|---|---|---|---|---|
| 1 | 350 | 39 | 9 | 85.9195 | 963.0675 |
| 2 | 298 | 26 | 8 | 79.2796 | 663.5594 |
| 3 | 422 | 31 | 7 | 85.4688 | 665.5167 |
| 4 | 281 | 16 | 9 | 70 | 665 |
| 5 | 301 | 16 | 6 | 75 | 445 |
| 6 | 360 | 29 | 9.1885 | 84.2709 | 1086.3847 |
| 7 | 332.8333 | 18 | 10 | 81.8333 | 562.6709 |
| 8 | 276 | 33 | 5 | 78 | 590 |
| 9 | 323 | 25 | 5 | 75 | 1074 |
| 10 | 444 | 64 | 6 | 104 | 1199 |
| 11 | 323 | 25 | 5 | 75 | 1074 |
| 12 | 444 | 64 | 6 | 104 | 1199 |

**Table 6. The results of fixed cost allocation with trade-offs 2.**

| DMUs | Scale of Data | $\eta_j$ | $\tilde{r}_j$ | Fixed Cost Allocation |
|---|---|---|---|---|
| 1 | 417496.794 | 0.091549654 | 9.154965361 | 9.0323 |
| 2 | 246620.3301 | 0.054079471 | 5.407947107 | 7.4578 |
| 3 | 345458.6895 | 0.075752973 | 7.5752973 | 8.9572 |
| 4 | 224910 | 0.04931878 | 4.931878014 | 0 |
| 5 | 167960 | 0.036830654 | 3.683065365 | 8.4149 |
| 6 | 475294.9538 | 0.104223767 | 10.42237665 | 10.2269 |
| 7 | 341523.0734 | 0.074889962 | 7.488996208 | 8.6774 |
| 8 | 209752 | 0.045994899 | 4.599489917 | 6.4 |
| 9 | 405597 | 0.08894024 | 8.894023952 | 10.427 |
| 10 | 658812.2134 | 0.144465852 | 14.44658517 | 9.9898 |
| 11 | 397164.7165 | 0.087091189 | 8.709118907 | 10.427 |
| 12 | 669742 | 0.14686256 | 14.68625604 | 9.9898 |
| sum | 4560331.771 | - | - | 100 |

# 5 | Conclusion

In this paper, we presented a fixed cost allocation plan considering production trade-offs among inputs and outputs. In this approach, we employed the strategy of maintaining unchanged efficiency of units after cost allocation. Under this principle, the efficiency of units remains unchanged after the allocation of fixed costs. To account for the relative importance of inputs and outputs, we used the production trade-offs method in the cost allocation model. We also considered two factors in the cost allocation of units: the first factor being the efficiency of the units, and the second factor the size of the inputs and outputs. In the practical example section, we demonstrated that by selecting different production trade-off matrices, we can incorporate the relative importance of inputs and outputs in the cost allocation process, resulting in divergent outcomes. As future work, the proposed models can be developed for a two-stage network structure in DEA. Additionally, we can utilize other strategies, such as achieving unit efficiency after allocation.

# Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this article.

# Data Availability

All data generated or analyzed during this study are included in this published article. No additional data are available.

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