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Efficiency Analysis of DMUs based on Separation Hyperplanes in PPS with CRS Technology to Deal with Interval Scale Data

M. Mohammadpour*

Department of Mathematics, Boukan Branch, Islamic Azad University, Boukan, Iran;

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Abstract

In this paper, in order to evaluate the performance of a Decision Making Unit (DMU) in a Production Possible Set (PPS) with Constant-to-Scale (CRS) technology, we provide models to obtain nonnegative weights for inputs and outputs which for the weights, the number of which DMUs for each one its virtual output does not exceed (is less than, if any) its virtual input be maximum, provided that for DMU under evaluation, the virtual output will be equal to the virtual input and the virtual input will be positive. We call these weights the relatively best weight (the relatively strongest weight, if any) for the DMU under evaluation, and if all the weights are positive, we call them the best weight (the strongest weight, if any) for the DMU under evaluation. Also, we define efficiency and strictly efficiency (strongly efficiency), respectively, as the ratio of the number of DMUs for each one for the relatively best weight and the best weight (the relatively strongest weight); its virtual input does not exceed (is less) its virtual input, to the total DMUs. The relatively best weight in input-oriented (the relatively strongest weight, if any) indicates the normal vector of a surface in the PPS with CRS assumption that the DMU under evaluation is on the surface and the maximum number of which DMUs their performance is no worse than (is better than) the DMU under evaluation separate from the rest of DMUs, with the constraint that the virtual input be positive. Accordingly, it can be interpreted the rest of the definitions of non-negative weights for inputs and outputs based on separation hyperplanes. Also, in this paper, we present the relationship between these definitions of efficiency with efficiency in the DEA models with constant returns to scale assumption.

Keywords: Data envelopment analysis, Efficiency analysis, Separation hyperplanes.

1 | Introduction

The CCR Charnes et al. [1] ratio form of Data Envelopment Analysis (DEA) obtains nonnegative weights for inputs and outputs by maximizing the ratio of virtual output with to virtual input, provided that the ratio does not exceed one for each Decision Making Unit (DMU) [2], [3]. Applying the Charnes and Cooper [4] theory of fractional programming, we can convert the CCR ratio form to the CCR multiplier form. The CCR

Corresponding Author:
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multiplier form obtains negative weights for inputs and outputs by maximizing the virtual output, provided that the virtual output does not exceed the virtual input for each DMU and the virtual output is equal to one.

Now, if in the CCR multiplier form, we replace constraint "the virtual output of DMU under evaluation equal to one" with "the virtual output of DMU under evaluation greater than zero", then we can say that the derived CCR multiplier form obtains the nonnegative values for the input weights and the output weights by maximizing the virtual output, provided that the virtual output does not exceed the virtual input for each DMU and the virtual output be greater than zero.

In other words, the derived CCR multiplier form obtains nonnegative weights for the inputs and the outputs of the DMU under evaluation by maximizing the income resulting from the outputs of the DMU, provided that the income resulting from the outputs of each DMU does not exceed the cost resulting from the inputs, and the cost resulting from the inputs of the DMU under evaluation be positive. Thus, the derived CCR multiplier form evaluates DMU under evaluation in the best conditions.

In this paper, with respect to one's inspiration from the derived CCR multiplier form, to evaluate the performance of a DMU in comparison with a set of DMUs, we obtain nonnegative weights for inputs and outputs which for the weights, the number of which DMUs for each one its virtual output does not exceed (is less than, if any) its virtual input be maximum, provided that for DMU under evaluation, the virtual output will be equal to the virtual input and the virtual input will be positive.

In other words, we are going to obtain negative weights for the inputs and the outputs of DMU under evaluation per weighs, the number of which DMUs for each one its income does not exceed (is less than, if any) its cost be maximum, provided that for DMU under evaluation, its income will be equal to its cost and cost resulting from the inputs of the DMU will be positive [5]. We call these weights the relatively best weight (the relatively strongest weight, if any) for the DMU under evaluation, and if all the weights are positive, we call them the best weight (the strongest weight, if any) for the DMU under evaluation.

Also, we define efficiency and strict efficiency (strong efficiency), respectively, as the ratio of the number of DMUs for the relatively best weight and the best weight (the relatively strongest weight); those virtual inputs do not exceed (is less) those virtual input, to the total DMUs [6]. The relatively best weight (if any, the relatively strongest weight) indicates the normal vector of a surface in the Production Possible Set (PPS) with returns to scale constant assumption that the DMU under evaluation is on the surface and the maximum number of which DMUs their performance are no worse than (is better than) the DMU under evaluation separate from the rest of DMUs [7], [8]. In this paper, we present the relationship between these definitions of efficiency with efficiency in the DEA models with constant returns to scale assumption [9], [10].

2 | Preliminaries

Suppose we have $n \geq 2$ peer observed DMUs,$\{DMU_j: j = 1,2, \dots, n\}$ which produce multiple outputsy_{rj}, $(r = 1, \dots, s)$, by utilizing multiple inputs x_{ij}, $(i = 1, \dots, m)$.The input and output vectors of DMU_j are denoted by x_j and y_j, respectively, and we assume that x_j and y_j are semipositive, i.e. ,$x_j \geq 0,\ x_j \neq 0$ and $y_j \geq 0,\ y_j \neq 0$ for $i = 1, \dots, n$. We use by (x_j, y_j) to descript DMU_j, and specially use $(x_o,\ y_o)(o\epsilon\{1,2, \dots, n\})$ as the DMU under evaluation. Throughout this paper, vectors will be denoted by bold letters.

2.1 | The Charnes, Cooper and Rhodes Model

The production set P_c of the CCR model [1] is defined as a set of semi-positive (x, y) as follows:

$$P_c = \left\{(x, y) | x \geq \sum_{j=1}^{n} \lambda_j x_j \& y \leq \sum_{j=1}^{n} \lambda_j y_j \& \lambda_j \geq 0 \ \ j = 1, \dots, n\right\},$$

where $(\lambda_1, \dots, \lambda_n)$ is a semipositive in $\mathbb{R}^n$.The input-oriented CCR model evaluates the efficiency of each DMU_o by solving the following linear program:

$$\begin{aligned}
\theta^* \quad & = \min \theta, \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_j \le \theta x_o \,(a), \\
& \sum_{j=1}^{n} \lambda_j y_j \ge y_o \,(b), \\
& \lambda_j \ge 0, \;\; j = 1, \dots, n,
\end{aligned} \tag{1}$$

where θ is a scaler. Because x_j and y_j are semipositive for $j = 1,2, \dots, n$, $\theta^* > 0$. Also since $(\theta, \lambda = (\lambda_1, \dots, \lambda_n)$ is a feasible solution to *Model (1)*, where $\theta = 1$, $\lambda_j = 0 (j \neq o), \lambda_o = 1$, then $\theta^* \le 1$. Thus $0 < \theta^* \le 1$. θ^*represents the input-oriented CCR-efficiency value of DMU_o.

Definition 1. (Radial efficiency): The performance of DMU_o is radial efficient if and only if $\theta^* = 1$.

The dual problem of *Model (1)* is expressed as:

$$\begin{aligned}
z^* \quad & = \max \;\; u^t y_o, \\
\text{s.t.} \quad & v^t x_o = 1, \\
& u^t y_j \le v^t x_j, \qquad j = 1,2, \dots, n. \\
& u \ge o, v \ge o,
\end{aligned} \tag{2}$$

where $v \epsilon \mathbb{R}^m$ and $u \epsilon \mathbb{R}^s$ are row vectors and represent dual variables corresponding to (a) and (b), respectively.From strong duality theorem $\theta^* = z^*$, thus $0 < z^* \le 1$

2.2 | The Two Phases of the Charnes, Cooper and Rhodes Model

The two-phase process for the CCR model evaluates the efficiency of DMU_o by solving the following linear program:

$$\begin{aligned}
\min \quad & \theta - \varepsilon \Big(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+ \Big), \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io}, \qquad i = 1, \dots, m (c), \\
& \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro}, \qquad r = 1, \dots, s (d), \\
& \lambda_j \ge 0, s_i^- \ge 0, s_r^+ \ge 0, \;\; \text{for all } j, \text{for all } i, \text{for all } r,
\end{aligned} \tag{3}$$

where $\varepsilon > 0$ is the non-Archimedian element. The presence of a non-Archimedean element, ε, in the function of *Model (2)* effectively allows the minimization over θ to preempt the optimization involving the slacks

$s^- = (s_1^-, \dots, s_m^-), s^+ = (s_1^+, \dots, s_s^+)$.

Definition 2. (CCR-efficient). The performance of DMU_o is CCR-efficient if only if an optimal solution $((\theta^*, \lambda^*, s^{*-}, s^{*+})$of the two-phase *Model (2)* satisfies $\theta^* = 1, s^{-*} = 0, s^{+*} = 0$.

The dual multiplier form of the program *Model (2)* is expressed as:

$$\begin{aligned}
\max \quad & u^t y_o, \\
\text{s.t.} \quad & v^t x_o = 1, \\
& u^t y_j \le v^t x_j, \;\; \text{for all } j, \\
& u \ge 1\varepsilon, v \ge 1\varepsilon.
\end{aligned} \tag{4}$$

Definition 3. The performance of $\mathrm{DMU_o}$ is fully efficient if only if an optimal solution (u^*, v^*) of *Model (2)* satisfies $u^t y_o = 1$.

Theorem 1. The CCR-efficient given in *Definition 2* is equivalent to that given by *Definition 3*.

Proof: See [11].

Definition 4. (Reference set) reference set of $\mathrm{DMU_o}$ denoted by E_o is defined as:

$$E_o = \left\{\mathrm{DMU_j} \middle| j\epsilon\{1, \dots, n\} \& \lambda_j^* > 0 \text{ insomeoptimalsolution}(\theta^*, \lambda^*, s^{-*}, s^{+*}) \text{ of model (3)}\right\}.$$

Theorem 2. The DMUs in E_o are CCR-efficient.

Proof: See [11].

Definition 5. (Extreme CCR-efficient) $\mathrm{DMU_o}$ is extreme CCR-efficient if only if $E_o = \{\mathrm{DMU_o}\}$.

Theorem 3. If $\mathrm{DMU_o}$ be extreme CCR-efficient, then $\mathrm{DMU_o}$ is CCR-efficient.

Proof: See [11].

Theorem 4. $\mathrm{DMU_o}$ is extreme CCR-efficient if

$$\begin{aligned}
\min \quad & \theta - \varepsilon \sum_{j \neq o} \lambda_{j,} \\
\text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_j \leq \theta x_{o,} \\
& \sum_{j=1}^{n} \lambda_j y_j \geq y_o\,, \\
& \lambda_j \geq 0, \quad \text{for all } j.
\end{aligned} \tag{5}$$

Has an optimal objective function value of one.

Proof: Let $\mathrm{DMU_o}$ not be extreme CCR-efficient. Then, there exists an optimal solution $(\theta^*, \lambda^*, s^{*-}, s^{*+})$ of *Model (2)* such that a $\lambda_j^* > 0 (j \neq o)$. Also, since $(\theta, \lambda = (\lambda_1, \dots, \lambda_n))$ is a feasible solution to *Model (4)*, where $\theta = 1$, $\lambda_j = 0 (j \neq o), \lambda_o = 1$, thus $\theta^* \leq 1$. Therefore $\theta^* - \varepsilon \sum_{j\neq o} \lambda^*{}_j < 1$. Let the solution objective function value of *Model (4)* be less than one, and let $(\tilde{\theta}, \tilde{\lambda})$ is an optimal solution of the model, then either $\tilde{\theta} < 1$ or $\tilde{\theta} = 1$ and $\sum_{j\neq o} \tilde{\lambda}_j > 0$). If $\tilde{\theta} < 1$, $\mathrm{DMU_o}$ is not extreme CCR-efficient. If $\tilde{\theta} = 1$ and $\sum_{j\neq o} \tilde{\lambda}_j > 0$, then either

$$(\tilde{s}^-, \tilde{s}^+) \neq (0,0),$$

Or

$$(\tilde{s}^-, \tilde{s}^+) = (0,0),$$

where

$$\tilde{s}^- = \tilde{\theta} x_o - \textstyle\sum_{j=1}^{n} \tilde{\lambda}_j x_j,$$

and

$$\tilde{s}^+ = \textstyle\sum_{j=1}^{n} \tilde{\lambda}_j y_j - y_o.$$

If $(\tilde{s}^-, \tilde{s}^+) \neq (0,0)$, since $(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+)$ is a feasible solution of *Model (2)*, thus $\mathrm{DMU_o}$ isn't CCR-efficient, therefore $\mathrm{DMU_o}$ is not extreme CCR-efficient. If $(\tilde{s}^-, \tilde{s}^+) = (0,0)$, then either $(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+)$ is an optimal solution of *Model (2)* or isn't. If $(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+)$ be an optimal solution of *Model (2)*, since $\sum_{j=1}^{n} \tilde{\lambda}_j > 0$, thus $\mathrm{DMU_o}$ is not extreme CCR-efficient. If $(\tilde{\theta}, \tilde{\lambda}, \tilde{s}^-, \tilde{s}^+)$ not be an optimal solution of *Model (2)*, then there exists an optimal solution $(\theta^*, \lambda^*, s^{*-}, s^{*+})$ of *Model (2)* such that $\theta^* = 1$ and $(s^{*-}, s^{*+}) \neq (0,0)$, thus $\mathrm{DMU_o}$ is not extreme CCR-efficient.

3 | Efficiency Analysis of Decision Making Units-based Separation Hyperplanes in Production Possible Set with Constant-to-Scale Technology

Definition 6. Let $\Lambda_c \subset \mathbb{R}^{m+s}$ be

$$\Lambda_c = \{(u,v) | u \in \mathbb{R}^s \& v \in \mathbb{R}^m \& (u,v) \geq (0,0)\}. \tag{6}$$

We define a map

$$f_c: \Lambda_c \to \mathbb{N} \cup \{0\}.$$

By

$$f_c(u,v) = \left| \left\{ \mathrm{DMU_j} \middle| j \in \{1,2,\dots,n\} \& v^t x_j \geq u^t y_j \right\} \right|, \tag{7}$$

where symbol $|\,.\,|$ is the cardinality of sets.

Definition 7. We define a map

$$g_c: \Lambda_c \to \mathbb{N} \cup \{0\}.$$

By

$$g_c(u,v) = \left| \left\{ \mathrm{DMU_j} \middle| j \in \{1,2,\dots,n\} \& v^t x_j > u^t y_j \right\} \right|, \tag{8}$$

where Λ_c is defined by *Model (6)*.

Definition 8. Let $(u_o, v_o) \in \Lambda_c$. We say (u_o, v_o) is a relatively best weight in Λ_c for $\mathrm{DMU_o}$ if

$$v_o^t x_o = u_o^t y_o \& u_o^t y_o > 0,$$

and

$$\text{for all } (u,v)\big((u,v) \in \Lambda_c \& u^t y_o > 0 \ \& v^t x_o = u^t y_o \implies f_c(u_o, v_o) \geq f_c(u,v)\big).$$

Definition 9. Let $(u_o, v_o) \in \Lambda_c$. We say (u_o, v_o) is the best weight in Λ_c for $\mathrm{DMU_o}$ if

$$v_o^t x_o = u_o^t y_o \& (u_o, v_o) > (0,0),$$

and

$$\text{for all } (u,v)\big((u,v) \in \Lambda_c \& (u,v) > (0,0) \& v^t x_o = u^t y_o \implies f_c(u_o, v_o) \geq f_c(u,v)\big).$$

Definition 10. Let $(u_o, v_o) \in \Lambda_c$. We say (u_o, v_o) is relatively strongest weight in Λ_c for $\mathrm{DMU_o}$ if

$$v_o^t x_o = u_o^t y_o \& u_o^t y_o > 0 \ \& g_c(u_o, v_o) \geq 1,$$

and

$$\text{for all } (u,v)\big((u,v) \in \Lambda_c \& u^t y_o > 0 \ \& v^t x_o = u^t y_o \implies g_c(u_o, v_o) \geq g_c(u,v)\big).$$

Definition 11. Let $(u_o, v_o) \in \Lambda_c$. We say (u_o, v_o) is the strongest weight in Λ_c for $\mathrm{DMU_o}$ if

$$v_o^t x_o = u_o^t y_o \& (u_o, v_o) > (0,0) \ \& g_c(u_o, v_o) \geq 1,$$

and

$$\text{for all } (u,v)\big((u,v)\epsilon\Lambda_c\&(u,v)>(0,0)\&v^t x_o=u^t y_o \Longrightarrow g_c(u_o,v_o)\geq g_c(u,v)\big).$$

Remark 1. Since x_j and y_j are semi-positive, it follows

$$\textstyle\sum_{r=1}^{s} y_{ro}>0, \sum_{i=1}^{m} x_{io}>0.$$

Now if $\sum_{r=1}^{s} y_{ro}=\sum_{i=1}^{m} x_{io}$, then, by taking $v^t=(1,\dots,1)\epsilon\mathbb{R}^m$, and $u^t=(1,\dots,1)\epsilon\mathbb{R}^s$, we have

$$v^t x_o-u^t y_o, u^t y_o>0, u\geq 1\varepsilon, v\geq 1\varepsilon.$$

If $\sum_{r=1}^{s} y_{ro}>\sum_{i=1}^{m} x_{io}$, then, by taking

$$\alpha=\left(\sum_{r=1}^{s} y_{ro}\Big/\sum_{i=1}^{m} x_{io}\right), u^t=\alpha(1,\dots,1)\epsilon\mathbb{R}^s,$$

and,

$$v^t=\alpha(1,\dots,1)\epsilon\mathbb{R}^m,$$

We have

$$v^t x_o-u^t y_o, u^t y_o>0, u\geq 1\varepsilon, v\geq 1\varepsilon.$$

Finally if $\sum_{r=1}^{s} y_{ro}<\sum_{i=1}^{m} x_{io}$ then, by taking

$$\beta=\left(\sum_{i=1}^{m} x_{io}\Big/\sum_{r=1}^{s} y_{ro}\right),$$

$$u^t=\beta(1,\dots,1)\epsilon\mathbb{R}^s,$$

and

$$v^t=\beta(1,\dots,1)\epsilon\mathbb{R}^m,$$

We have

$$v^t x_o-u^t y_o,\ u^t y_o>0,\ u\geq 1\varepsilon,\ v\geq 1\varepsilon.$$

This shows that there is not any relatively strongest weight in Λ_c for $\mathrm{DMU_o}$ if

$$\text{for all } (u,v)\big((u,v)\epsilon\Lambda_c\&u^t y_o>0\ \&v^t x_o=u^t y_o \Longrightarrow g_c(u,v)=0\big).$$

Also, there is no strongest weight in Λ_c for $\mathrm{DMU_o}$ if

$$\text{for all } (u,v)\big((u,v)\epsilon\Lambda_c\&(u,v)>(0,0)\&v^t x_o=u^t y_o \Longrightarrow g_c(u,v)=0\big).$$

Definition 12. (Λ_c-efficiency) If (u_o,v_o) be relatively best weight in Λ_c for $\mathrm{DMU_o}$, then

Λ_c-efficiency of

$$\mathrm{DMU_o}=\frac{f_c(u_o,v_o)}{n}$$

Definition 13. (Λ_c-efficient) $\mathrm{DMU_o}$ is said to be Λ_c-efficient if Λ_c-efficiency of $\mathrm{DMU_o}$ = 1.

Definition 14. (Strictly Λ_c-efficiency) If (u_o,v_o) be the best weight in Λ_c, for $\mathrm{DMU_o}$ then strictly

$$\Lambda_c\text{- efficiency}=\frac{f_c(u_o,v_o)}{n}.$$

Definition 15. (Strictly Λ_c-efficient) $\mathrm{DMU_o}$ is said to be strictly Λ_c-efficient if strictly Λ_c-efficiency of $\mathrm{DMU_o}$ = 1.

Definition 16. (Strongly Λ_c-efficiency) If there is a (u_o, v_o) relatively strongest weight in Λ_c for DMU_o. Then, strongly

$$\Lambda_c\text{-efficiency} = \frac{g_c(u_o,v_o)}{n-1}.$$

Definition 17. If there is not any relatively strongest weight in Λ_c for DMU_o, then strongly Λ_c-efficiency =0.

Definition 18. (Strongly Λ_c-efficient) DMU_o is said to be strongly Λ_c-efficient if strongly Λ_c-efficiency of $DMU_o = 1$.

Proposition 1. Let $(\bar{u}, \bar{v})$ be relatively best weight in Λ_c for DMU_o, and let $(\bar{u}, \bar{v}) > (0,0)$. Then $(\bar{u}, \bar{v})$ is the best weight in Λ_c for DMU_o.

Proof: If $(\bar{u}, \bar{v})$ not be the best weight in Λ_c for DMU_o, then by *Definition 4*, there is some $(\tilde{u}, \tilde{v}) \epsilon \Lambda_c$ such that

$$(\tilde{u}, \tilde{v}) > (0,0),\ \tilde{v}^t x_o = \tilde{u}^t y_o,$$

and

$$f_c(\bar{u}, \bar{v}) < f_c(\tilde{u}, \tilde{v}),$$

Thus, since x_j and y_j are semi-positive, $\tilde{u}^t y_o (= \tilde{v}^t x_o) > 0$. Therefore, noting that $(\bar{u}, \bar{v})$ is relatively best weight in Λ_c for DMU_o,. $f_c(\bar{u}, \bar{v}) \geq f_c(\tilde{u}, \tilde{v})$ which is in contradiction with the fact that $f_c(\bar{u}, \bar{v}) < f_c(\tilde{u}, \tilde{v})$. Thus $(\bar{u}, \bar{v})$ is the best weight in Λ_c for DMU_o.

Proposition 2. Let (u_o, v_o) be relatively strongest weight in Λ_c for DMU_o, and let $(u_o, v_o) > (0,0)$. Then (u_o, v_o) is the strongest weight in Λ_c for DMU_o.

Proof: Similar to the proof of *Theorem 2*.

Theorem 5. Let $(\bar{u}, \bar{v})$ be relatively best weight in Λ_c for DMU_o, let $p = f_c(\bar{u}, \bar{v})$

$$\left\{DMU_{j_1}, \dots, DMU_{j_p}\right\} = \left\{DMU_j \,\middle|\, j\epsilon\{1, \dots, n\} \;\&\; \bar{v}^t x_o \geq \bar{u}^t y_o\right\}, (a_1).$$

Letand let $\bar{t} = (\bar{t}_1, \dots, \bar{t}_n)$ with

$$\bar{t}_j = \begin{cases} 0 & j\epsilon\{j_1, \dots, j_p\}, \\ 1 & j\epsilon\{1, \dots, n\} - \{j_1, \dots, j_p\}. \end{cases} (a_2)$$

Then $(\bar{u}, \bar{v}, \bar{t})$ is an optimal solution for the following model

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} t_j, \\ \text{s.t.} \quad & v^t x_o - u^t y_o = 0,\ u^t y_o \geq \varepsilon, \\ & v^t x_j - u^t y_j + M t_j \geq 0,\ \text{for all } j,\ (M \gg 0) \\ & u \geq 0,\ v \geq 0,\ t_j \epsilon \{0,1\}. \end{aligned} \tag{9}$$

If $(\tilde{u}, \tilde{v}, \tilde{t})$ be an optimal solution of *Model (9)*, then $(\tilde{u}, \tilde{v})$ is a relatively best weight in Λ_c for DMU_o.

Proof: Since $(\bar{u}, \bar{v})$ is a relatively best weight in Λ_c for DMU_o, then, by (a_1) and (a_2), $(\bar{u}, \bar{v}, \bar{t})$ is a feasible solution to *Model (9)*. Also, since $(\tilde{u}, \tilde{v}, \tilde{t})$ is an optimal solution for *Model (9)*, we have

$$n - \sum_{j=1}^{n} \tilde{t}_j \geq n - \sum_{j=1}^{n} \bar{t}_j.$$

Therefore

$$f_c(\tilde{u},\tilde{v}) \geq f_c(\bar{u},\bar{v}),$$

and

$$f_c(\tilde{u},\tilde{v}) \leq f_c(\bar{u},\bar{v}).$$

Since $(\bar{u},\bar{v})$ is relatively best weight in Λ_c for DMU_o. Hence,

$$f_c(\tilde{u},\tilde{v}) = f_c(\bar{u},\bar{v}) = n - \sum_{j=1}^{n} \bar{t}_j = n - \sum_{j=1}^{n} \tilde{t}_j.$$

Thus, $(\bar{u},\bar{v},\bar{t})$ is an optimal solution to *Model (9)*, and $(\tilde{u},\tilde{v})$ is relatively best weight in Λ_c for DMU_o.

Theorem 6. Let $(\bar{u},\bar{v},\bar{t})$ be an optimal solution for the following model

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} t_j, \\ \text{s.t.} \quad & v^t x_o = 1, \; u^t y_o = 1, \\ & v^t x_j - u^t y_j + M t_j \geq 0, \; \text{for all } j, \; (M \gg 0) \\ & u \geq 0, \; v \geq 0, \; t_j \epsilon \{0,1\}. \end{aligned} \tag{10}$$

Then, $(\bar{u},\bar{v})$ is relatively best weight in Λ_c for DMU_o.

Proof: Let $(\tilde{u},\tilde{v})$ be relatively best weight in Λ_c for DMU_o, let $p = f_c(\tilde{u},\tilde{v})$ with

$$\left\{DMU_{j_1}, \dots, DMU_{j_p}\right\} = \left\{DMU_j \middle| j\epsilon\{1,\dots,n\} \& \tilde{v}^t x_o \geq \tilde{u}^t y_o\right\},$$

Then

$$\tilde{v}^t x_o = \tilde{u}^t y_o, \tilde{u}^t y_o > 0,$$

and

$$\tilde{v}^t x_{j_i} - \tilde{u}^t y_{j_i} \geq 0, i = 1, \dots, p.$$

So that, by taking $k = \tilde{u}^t y_o$, $\hat{u} = {}^{\tilde{u}}/_{k}$, and $\hat{v} = {}^{\tilde{v}}/_{k}$, we have

$$(\hat{u},\hat{v}) \geq (0,0), \; \hat{v}^t x_o = \hat{u}^t y_o = 1, \hat{v}^t x_{j_i} - \hat{u}^t y_{j_i} \geq 0, \; i = 1, \dots, p. \; (a_3).$$

Thus $(\hat{u},\hat{v},\hat{t})$ is a feasible solution for *Model (10)*, where $\hat{t} = (\hat{t}_1, \dots, \hat{t}_n)$ with

$$\hat{t}_j = \begin{cases} 0 & j\epsilon\{j_1, \dots, j_p\}, \\ 1 & j\epsilon\{1,\dots,n\} - \{j_1,\dots,j_p\}. \end{cases}$$

Therefore, $(\bar{u},\bar{v},\bar{t})$ is an optimal solution for *Eq. (10)*,

$$n - \sum_{j=1}^{n} \hat{t}_j \leq n - \sum_{j=1}^{n} \bar{t}_j).$$

Hence

$$f_c(\hat{u},\hat{v}) \leq f_c(\bar{u},\bar{v}) \; (a_4),$$

by

$$(a_4), p \leq f_c(\hat{u},\hat{v}).$$

Thus, since $(\tilde{u},\tilde{v})$ is relatively best weight in Λ_c for DMU_o,

$$p = f_c(\tilde{u},\tilde{v}) = f_c(\hat{u},\hat{v}). (a_5).$$

Also, since $(\tilde{u},\tilde{v})$ is relatively best weight in Λ_c for DMU_o, we have $f_c(\tilde{u},\tilde{v}) \geq f_c(\bar{u},\bar{v})$, then, by (a_3), (a_4), and (a_5), $f_c(\tilde{u},\tilde{v}) = f_c(\bar{u},\bar{v})$, Thus $(\bar{u},\bar{v})$ is relatively best weight in Λ_c for DMU_o.

Corollary 1. Let $(\bar{u},\bar{v},\bar{t})$ be an optimal solution of *Model (10)*, then

Λ_c-efficiency of $DMU_o = \frac{f_c(\bar{u},\bar{v})}{n} = \frac{n-\sum_{j=1}^{n}\bar{t}_j}{n}$.

Proof: *Theorem (6).*

Corollary 2. DMU_o is Λ_c-efficient if and only if the optimal objective function value of *Model (10)* is zero.

Proof: *Theorem (6).*

Corollary 3. If there is some optimal solution $(\bar{u},\bar{v},\bar{t})$ for *Model (10)* such that $(\bar{u},\bar{v}) > (0,0)$, then $(\bar{u},\bar{v})$ is the best weight in Λ_c for DMU_o.

Proof: *Proposition 1* and *Theorem (7).*

Corollary 4. If there is some optimal solution $(\bar{u},\bar{v},\bar{t})$ for *Model (10)* such that $(\bar{u},\bar{v}) > (0,0)$, and $\sum_{j=1}^{n}\bar{t}_j = 0$, then DMU_o is strictly Λ_c-efficient.

Proof: *Proposition 1*and *Theorem (7).*

Theorem 7. Let $(\tilde{u},\tilde{v},\tilde{t})$ be an optimal solution for the following model

$$
\begin{aligned}
\min \quad & \sum_{j=1}^{n} t_j, \\
\text{s.t.} \quad & v^t x_o - u^t y_o = 0, \\
& v^t x_j - u^t y_j + M t_j \geq 0, \quad \text{for all } j, \quad (M \gg 0) \\
& u \geq 1\varepsilon, v \geq 1\varepsilon, \ t_j \epsilon \{0,1\}.
\end{aligned}
\tag{11}
$$

Then $(\tilde{u},\tilde{v})$ is the best weight in Λ_c for DMU_o.

Proof: First, let us show *Model (11)* is feasible. Since x_j and y_j are semi-positive, it follows that

$\sum_{r=1}^{s} y_{ro} > 0, \ \sum_{i=1}^{m} x_{io} > 0.$

Now, if $\sum_{r=1}^{s} y_{ro} = \sum_{i=1}^{m} x_{io}$, then, by taking $v^t = (1,\ldots,1)\epsilon\mathbb{R}^m$, and $u^t = (1,\ldots,1)\epsilon\mathbb{R}^s$, we have

$v^t x_o - u^t y_o, \ u \geq 1\varepsilon, \ v \geq 1\varepsilon.$

If

$\sum_{r=1}^{s} y_{ro} > \sum_{i=1}^{m} x_{io},$

Then, by taking

$\alpha = (\sum_{r=1}^{s} y_{ro} / \sum_{i=1}^{m} x_{io}), u^t = \alpha(1,\ldots,1)\epsilon\mathbb{R}^s,$

and

$v^t = \alpha(1,\ldots,1)\epsilon\mathbb{R}^m,$

we have

$v^t x_o - u^t y_o, \ u \geq 1\varepsilon, \ v \geq 1\varepsilon.$

If

$$\sum_{r=1}^{s} y_{ro} < \sum_{i=1}^{m} x_{io},$$

Then, by taking

$$\beta = \left(\sum_{i=1}^{m} x_{io} \middle/ \sum_{r=1}^{s} y_{ro}\right),\ u^t = \beta(1, \dots, 1) \epsilon \mathbb{R}^s,$$

and

$$v^t = \beta(1, \dots, 1) \epsilon \mathbb{R}^m,$$

we have

$$v^t x_o - u^t y_o,\ u \geq 1\varepsilon,\ \ v \geq 1\varepsilon.$$

Thus (u, v, t), where t is defined by (a_1) and (a_2), is a feasible solution for *Model (11)*.

Let $(\bar{u}, \bar{v})$ be the best weight in Λ_c for DMU_o, let $p = f_c(\bar{u}, \bar{v})$, and let

$$\left\{DMU_{j_1}, \dots, DMU_{j_p}\right\} = \left\{DMU_j \middle| j \epsilon \{1, \dots, n\} \& \bar{v}^t x_o \geq \bar{u}^t y_o\right\},$$

Then

$$(\bar{u}, \bar{v}) \geq (0,0),\ \ \bar{v}^t x_o = \bar{u}^t y_o, \bar{v}^t x_{j_i} - \bar{u}^t y_{j_i} \geq 0,\ \ i = 1, \dots, p.$$

Taking

$$k = \min\left\{\bar{u}^t y_o, \min_r\{\bar{u}_r\}, \min_i\{\bar{v}_i\}\right\},$$

We have

$$(\bar{u}, \bar{v}) \geq k(1,1),\ \ \bar{v}^t x_o = \bar{u}^t y_o,\ \ \bar{u}^t y_o \geq k(>0), \bar{v}^t x_{j_i} - \bar{u}^t y_{j_i} \geq 0,\ \ i = 1, \dots, p.$$

Thus $(\bar{u}, \bar{v}, \bar{t})$, where $\bar{t}$ is defined by (a_2), is a feasible solution for *Model (11)*, and since $(\tilde{u}, \tilde{v}, \tilde{t})$ is an optimal solution of *Model (11)*; therefore,

$$n - \textstyle\sum_{j=1}^{n} \tilde{t}_j \geq n - \sum_{j=1}^{n} \bar{t}_j.$$

Hence $f_c(\tilde{u}, \tilde{v}) \geq f_c(\bar{u}, \bar{v})$. Also $f_c(\tilde{u}, \tilde{v}) \leq f_c(\bar{u}, \bar{v})$, since $(\bar{u}, \bar{v})$ is the best weight in Λ_c for DMU_o. Thus

$$f_c(\tilde{u}, \tilde{v}) = f_c(\bar{u}, \bar{v}) = n - \textstyle\sum_{j=1}^{n} \bar{t}_j = n - \sum_{j=1}^{n} \tilde{t}_j.$$

Then it follows that also $(\tilde{u}, \tilde{v})$ is the best weight in Λ_c for DMU_o, and $(\bar{u}, \bar{v}, \bar{t})$ is an optimal solution of *Model (11)*.

Theorem 8. Let $(\tilde{u}, \tilde{v}, \tilde{t})$ be an optimal solution for the following model

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} t_j, \\ \text{s.t.} \quad & v^t x_o - u^t y_o = 0, \\ & v^t x_j - u^t y_j + M t_j \geq 0,\ \ \text{for all } j,\ \ (M \gg 0) \\ & u \geq 1, v \geq 1,\ t_j \epsilon \{0,1\}. \end{aligned} \tag{12}$$

Then $(\tilde{u}, \tilde{v})$ is the best weight in Λ_c for DMU_o.

Proof: Similar to the proof of *Theorem 5*.

Corollary 5. Let $(\bar{u}, \bar{v}, \bar{t})$ be an optimal solution of *Model (12)*, then strictly

Λ_c-efficiency of $DMU_o = \frac{f_c(\bar{u},\bar{v})}{n} = \frac{n-\sum_{j=1}^{n} \bar{t}_j}{n}$.

Proof: Follows from *Theorem (8)*.

Corollary 6. DMU_o is strictly Λ_c-efficient if only if *Model (12)* has an optimal objective function value of zero.

Proof: Follows from *Theorem (8)*.

Theorem 9. Let $(\tilde{u}, \tilde{v}, \tilde{t})$ be an optimal solution for the following model

$$\begin{aligned} \min \quad & \sum_{j \neq o} t_j , \\ \text{s.t.} \quad & v^t x_o - u^t y_o = 0, v^t x_o \geq \varepsilon, \\ & v^t x_j - u^t y_j + M t_j \geq \varepsilon, \ \ j \neq o, \qquad (M \gg 0), \\ & u \geq 0, \ \ v \geq 0, \ t_j \epsilon \{0,1\}, j \neq o, \end{aligned} \tag{13}$$

where

$\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{o-1}, \tilde{t}_{o+1}, \dots, \tilde{t}_n)$.

If $\sum_{j \neq o} \tilde{t}_j < n - 1$, then $(\tilde{u}, \tilde{v})$ is a relatively strongest weight in Λ_c for DMU_o, but if $\sum_{j \neq o} \tilde{t}_j = n - 1$, then there is not any relatively strongest weight in Λ_c for DMU_o.

Conversely, if there is not any relatively strongest weight in Λ_c for DMU_o, then the optimal objective function value of *Model (13)* is equal to $n - 1$, but if there exists a $(\bar{u}, \bar{v}) \epsilon \Lambda_c$ such that $(\bar{u}, \bar{v})$ is relatively strongest weight in Λ_c for DMU_o, then by taking

$\left\{DMU_{j_1}, \dots, DMU_{j_p}\right\} = \{DMU_j | j \epsilon \{1, \dots, n\} \& \bar{v}^t x_j > \bar{u}^t y_j\} \ (a_6)$.

With $p = g_c(\bar{u}, \bar{v})$, and

$\bar{t} = (\bar{t}_1, \dots, \bar{t}_{o-1}, \bar{t}_{o+1}, \dots, \bar{t}_n)$.

With

$$\bar{t}_j = \begin{cases} 0 & j \epsilon \{j_1, \dots, j_p\}, \\ 1 & j \epsilon \{1, \dots, n\} - \{j_1, \dots, j_p, o\}. \end{cases} \ (a_7)$$

$(\bar{u}, \bar{v}, \bar{t})$ is an optimal solution for *Model (13)* and $\sum_{j \neq o} \bar{t}_j < n - 1$.

Proof: Let there be a $(\bar{u}, \bar{v}) \epsilon \Lambda_c$ such that $(\bar{u}, \bar{v})$ is relatively strongest weight in Λ_c for DMU_o, then $g_c(\bar{u}, \bar{v}) \geq 1$, so by taking that $p = g_c(\bar{u}, \bar{v})$ with

$\left\{DMU_{j_1}, \dots, DMU_{j_p}\right\} = \{DMU_j | j \epsilon \{1, \dots, n\} \& \bar{v}^t x_j > \bar{u}^t y_j\}$,

We have

$\bar{u} \geq 0, \bar{v} \geq 0, \bar{v}^t x_o = \bar{u}^t y_o , \bar{v}^t x_o > 0, \bar{v}^t x_{j_i} - \bar{u}^t y_i > 0 , \ \ i = 1, .., p.$

So by taking

$k = \min\left\{\bar{u}^t y_o, \min_i \{\bar{v}^t x_{j_i} - \bar{u}^t y_i\}\right\}$, We have $k > 0$ and

$$\bar{u} \geq 0, \bar{v} \geq 0, \bar{v}^t x_o = \bar{u}^t y_o, \bar{v}^t x_o \geq k, \bar{v}^t x_{j_i} - \bar{u}^t y_i \geq k, \quad i = 1, .., p.$$

Thus $(\bar{u}, \bar{v}, \bar{t})$ is a feasible solution for *Model (13)* where $\bar{t}$ is defined by (a_6) and (a_4), therefore, the optimal objective function value of *Model (13)* is less $n-1$. Also, since $(\tilde{u}, \tilde{v}, \tilde{t})$ is an optimal solution for the *Model (13)*,

$$n - 1 - \sum_{j\neq o} \tilde{t}_j \geq n - 1 - \sum_{j\neq o} \bar{t}_j.$$

It follows $g_c(\tilde{u}, \tilde{v}) \geq g_c(\bar{u}, \bar{v})$. On the other hand, $g_c(\tilde{u}, \tilde{v}) \leq g_c(\bar{u}, \bar{v})$, since $(\bar{u}, \bar{v})$ is relatively strongest weight in Λ_c for DMU_o. Consequently, $g_c(\tilde{u}, \tilde{v}) = g_c(\bar{u}, \bar{v})$. Thus $(\bar{u}, \bar{v}, \bar{t})$ is an optimal solution for *Model (13)*, and $(\tilde{u}, \tilde{v})$ is a relatively strongest weight in Λ_c for DMU_o. Also, it is easy to show if there is not any relatively strongest weight in Λ_c for DMU_o, then the optimal objective function value of *Model (13)* is equal to $n-1$.

Theorem 10. Let $(\tilde{u}, \tilde{v}, \tilde{t})$ be an optimal solution for the following model

$$\begin{array}{ll} \min & \sum_{j\neq o} t_j, \\ \text{s.t.} & v^t x_o - u^t y_o = 0, v^t x_o \geq 1, \\ & v^t x_j - u^t y_j + M t_j \geq 1, \ j \neq o, \qquad (M \gg 0), \\ & u \geq 0, v \geq 0, \ t_j \in \{0,1\}, j \neq o, \end{array} \tag{14}$$

Where

$$\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{o-1}, \tilde{t}_{o+1}, \dots, \tilde{t}_n).$$

If $\sum_{j\neq o} \tilde{t}_j < n-1$, then $(\tilde{u}, \tilde{v})$ is a relatively strongest weight in Λ_c for DMU_o, but if $\sum_{j\neq o} \tilde{t}_j = n-1$, then there is not any relatively strongest weight in Λ_c for DMU_o.

Proof: Similar to the proof of *Theorem 10*.

Corollary 7. Let $(\tilde{u}, \tilde{v}, \tilde{t})$ be an optimal solution for *Model (11)*, then strictly

$$\Lambda_c\text{-efficiency of } DMU_o = \frac{g_c(\tilde{u}, \tilde{v})}{n-1} = \frac{n - \sum_{j=1}^{n} \tilde{t}_j}{n-1}.$$

Proof: *Theorem (10)*.

Corollary 8. DMU_o is strongly Λ_c-efficiency if only if *Model (11)* has an optimal objective function value of zero.

Proof: *Theorem (10)*.

Theorem 11. Let $(\tilde{u}, \tilde{v}, \tilde{t})$ be an optimal solution for the following model

$$\begin{array}{ll} \min & \sum_{j\neq o} t_j, \\ \text{s.t.} & v^t x_o - u^t y_o = 0, \\ & v^t x_j - u^t y_j + M t_j \geq \varepsilon, \ j \neq o, \qquad (M \gg 0), \\ & u \geq 1\varepsilon, \ v \geq 1\varepsilon, \ t_j \in \{0,1\}, j \neq o, \end{array} \tag{15}$$

where

$$\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{o-1}, \tilde{t}_{o+1}, \dots, \tilde{t}_n).$$

If $\sum_{j\neq o}\tilde{t}_j < n-1$, then $(\tilde{u},\tilde{v})$ is a strongest weight in Λ_c for DMU_o, but if $\sum_{j\neq o}\tilde{t}_j = n-1$, then there is not any strongest weight in Λ_c for DMU_o.

If there is not any strongest weight in Λ_c for DMU_o, then the optimal objective function value of *Model (11)* is equal to $n-1$, but if there is a $(\bar{u},\bar{v})\epsilon\Lambda_c$ such that $(\bar{u},\bar{v})$ be the strongest weight in Λ_c for DMU_o, then $(\bar{u},\bar{v},\bar{t})$, where $\bar{t}$ is defined by (a_6) and (a_7), is an optimal solution for *Model (11)*, also $\sum_{j\neq o}\bar{t}_j < n-1$.

Proof: Similar to the proof of *Theorem 5*, it can be shown that *Model (16)* is feasible; also proof of the theorem is similar to the proof of *Theorem 7*.

Theorem 12. Let $(\tilde{u},\tilde{v},\tilde{t})$ be an optimal solution for the following model

$$\begin{aligned} \min \quad & \sum_{j\neq o} t_j, \\ \text{s.t.} \quad & v^t x_o - u^t y_o = 0, \\ & v^t x_j - u^t y_j + M t_j \geq 1, \ \ j\neq o, \qquad (M \gg 0). \\ & u \geq 1, \ \ v \geq 1, \ t_j \epsilon\{0,1\}, j\neq o. \end{aligned} \tag{16}$$

where

$$\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{o-1}, \tilde{t}_{o+1}, \dots, \tilde{t}_n).$$

If $\sum_{j\neq o}\tilde{t}_j < n-1$, then $(\tilde{u},\tilde{v})$ is a strongest weight in Λ_c for DMU_o, but if $\sum_{j\neq o}\tilde{t}_j = n-1$, then there is not any strongest weight in Λ_c for DMU_o.

The proof is similar to the proof of the *Theorem 7*.

Theorem 13. DMU_o is Λ_c-efficient if only if DMU_o is CCR-radial efficient.

Proof: Let DMU_o be Λ_c-efficient, then, by *Corollary 2*, the optimal objective function value of *Model (10)* is zero. Thus letting $(\bar{u},\bar{v},\bar{t})$ be an optimal solution for *Model (10)*, we have

$$\bar{u} \geq 0, \bar{v} \geq 0, \ \ \bar{v}^t x_o = \bar{u}^t y_o = 1, \bar{v}^t x_j - \bar{u}^t y_j \geq 0, \ \ j = 1,..,n.$$

Thus $(\bar{u},\bar{v})$ is an optimal solution for *Model (2)*. Therefore DMU_o is CCR-radial efficient. Conversely, let DMU_o be CCR-radial efficient, and let $(\tilde{u},\tilde{v})$ be an optimal solution of *Model (2)*, then

$$\tilde{u} \geq 0, \tilde{v} \geq 0, \ \ \tilde{v}^t x_o = \tilde{u}^t y_o = 1, \tilde{v}^t x_j - \tilde{u}^t y_j \geq 0, \ \ j = 1,..,n.$$

Thus $(\bar{u},\bar{v},\bar{t})$ where $\bar{t} = 0\epsilon\mathbb{R}^n$, is a feasible solution for *Model (2)*; therefore, the optimal objective function value of *Model (5)* is zero. Hence, by *Corollary 2*, DMU_o is Λ_c-efficient.

Theorem 14. DMU_o is strictly Λ_c- efficient if only if DMU_o is CCR-efficient.

Proof: Let DMU_o is strictly Λ_c- efficient. Then, by *Corollary 6*, the optimal objective function value *Model (7)* is zero, thus letting $(\bar{u},\bar{v},\bar{t})$ be an optimal solution for the model, we have

$$\bar{u} \geq 1, \bar{v} \geq 1, \ \ \bar{v}^t x_o = \bar{u}^t y_o, \bar{v}^t x_j - \bar{u}^t y_j \geq 0, \ \ j = 1,..,n.$$

Therefore, since x_o and y_o are semi-positive, it follows $\bar{v}^t x_o > 0, \ \bar{u}^t y_o > 0$. So by taking

$$k = \bar{u}^t y_o (= \bar{v}^t x_o), \tilde{u} = (\bar{u}/_k), \tilde{v} = (\bar{v}/_k), \alpha = (1/_k),$$

We have

$$\tilde{u} \geq 1\alpha, \ \ \tilde{v} \geq 1\alpha, \ \ \tilde{v}^t x_o = \tilde{u}^t y_o = 1, \tilde{v}^t x_j - \tilde{u}^t y_j \geq 0, \ \ j = 1,..,n.$$

Thus $(\tilde{u},\tilde{v})$ is a feasible solution for *Model (4)*, and since $\tilde{u}^t y_o = 1$, it follows that the optimal objective function value of *Model (9)* is equal to one. Hence, by *Corollary 6*, DMU_o is CCR–efficient.

Conversely, let DMU_o be CCR–efficient, then, by *Theorem 1*; there exists some optimal solution $(\hat{u},\hat{v})$ for *Model (4)* such that

$$\hat{u} > 0, \hat{v} > 0 \quad \hat{v}^t x_o = \hat{u}^t y_o = 1, \hat{v}^t x_j - \hat{u}^t y_j \geq 0\,, \quad j = 1,..,n.$$

Thus, by letting

$$k = \min\left\{\min_i\{\hat{v}_i\},\ \min_r\{\hat{u}_r\}\right\}, \bar{\bar{u}} = (\hat{u}/_k),$$

and $\bar{\bar{v}} = (\hat{v}/_k)$. We have $k > 0$ and

$$\bar{\bar{u}} \geq 1, \bar{\bar{v}} \geq 1,\ \bar{\bar{v}}^t x_o = \bar{\bar{u}}^t y_o, \bar{\bar{v}}^t x_j - \bar{\bar{u}}^t y_j \geq 0\,, \quad j = 1,..,n.$$

Therefore $(\bar{\bar{u}},\bar{\bar{v}},\bar{\bar{t}})$, where $\bar{\bar{t}} = 0 \epsilon \mathbb{R}^n$, is a feasible solution for *Model (7)*; hence, the optimal objective function value of the model is zero. Consequently, by *Corollary 6*, DMU_o is strictly Λ_c- efficient.

Theorem 15. DMU_o is strongly Λ_c- efficient if only if DMU_o is extreme CCR-efficient.

Proof: Let DMU_o be strongly Λ_c- efficient. Then, by *Corollary 8*, the optimal objective function value *Model (8)* is zero, thus letting $(\bar{u},\bar{v},\bar{t})$ be an optimal solution for the model where

$$\bar{t} = (\bar{t}_1, \dots, \bar{t}_{o-1}, \bar{t}_{o+1}, \dots, \bar{t}_n) = 0 \epsilon \mathbb{R}^{n-1},$$

We have

$$\bar{u} \geq 0,\ \bar{v} \geq 0,\ \bar{v}^t x_o - \bar{u}^t y_o = 0, \bar{v}^t x_o \geq 1,\ \bar{v}^t x_j - \bar{u}^t y_j \geq 1\,, \quad j = 1,..,n.$$

Therefore, by taking

$$k = \bar{v}^t x_o (= \bar{u}^t y_o), \tilde{u} = (\bar{u}/_k), \tilde{v} = (\bar{v}/_k), \text{ and } \alpha = (1/_k),$$

We have

$$\tilde{u} \geq 0,\ \tilde{v} \geq 0,\ \tilde{v}^t x_o = \tilde{u}^t y_o = 1, \tilde{v}^t x_j - \tilde{u}^t y_j \geq \alpha(> 0),\ j \neq o.$$

Thus $(\tilde{u},\tilde{v})$ is an optimal feasible solution for the following model

$$\begin{array}{ll} \max & u^t y_o, \\ \text{s.t.} & v^t x_o = 1, \\ & v^t x_o - u^t y_o \geq 0, \\ & v^t x_j - u^t y_j \geq \varepsilon\,,\ j \neq o, \\ & u \geq 0,\ v \geq 0. \end{array} \qquad \textbf{(16)}$$

Therefore, by the strong duality theorem, the optimal objective function value of *Model (5)*, which is the dual form of *Model (9)*, is equal to one. Hence, by *Theorem 2*, DMU_o is extreme CCR-efficient. Conversely, if DMU_o is extreme CCR-efficient, then, by *Theorem 2* and *Corollary 8*, DMU_o is strongly Λ_c- efficient.

Theorem 16. Let DMU_o be strongly Λ_c- efficient, then there exists some $(\bar{u},\bar{v}) \epsilon \Lambda_c$ such that $(\bar{u},\bar{v})$ is also a relatively strongest weight in Λ_c for DMU_o and a strongest weight in Λ_c for DMU_o.

Proof: Suppose $\mathrm{DMU_o}$ is strongly Λ_c- efficient, then, by *Theorem 14*, $\mathrm{DMU_o}$ is extreme CCR-efficient. Thus, by *Theorem 4*, *Theorem 3*, and *Definition 2*, it follows that the optimal objective function value of the following model is one

$$\begin{aligned} \min \quad & \theta - \varepsilon\Big(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_r^+\Big) - \varepsilon^2 \sum_{j\neq o} \lambda_j , \\ \text{s.t.} \quad & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = \theta x_{io} , i = 1, \dots, m, \\ & \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{ro} , r = 1, \dots, s, \\ & \lambda_j \geq 0, s_i^- \geq 0, s_r^+ \geq 0. \quad \text{for all } j, \text{for all } i, \text{for all } r, \end{aligned} \tag{17}$$

Therefore, by strong duality theorem, the optimal objective function value of the following model, which is the dual form of *Model (10)*, is equal to one

$$\begin{aligned} \max \quad & u^t y_o, \\ \text{s.t.} \quad & v^t x_o = 1, \\ & v^t x_o - u^t y_o \geq 0, \\ & v^t x_j - u^t y_j \geq \varepsilon , \quad j \neq o, \\ & u \geq 1\varepsilon, \quad v \geq 1\varepsilon. \end{aligned} \tag{18}$$

Hence letting $(\bar{u}, \bar{v})$ be an optimal solution for *Model (11)*, we have

$$\bar{u} \geq 1\varepsilon, \bar{v} \geq 1\varepsilon, \bar{u}^t y_o = \bar{v}^t x_o = 1, \bar{v}^t x_j - \bar{u}^t y_j \geq \varepsilon , \quad j \neq o.$$

Since $(\bar{u}, \bar{v}, \bar{t})$ where $\bar{t} = 0 \epsilon \mathbb{R}^{n-1}$ is an optimal solution also for *Model (8)* and for *Model (11)*, it follows $(\bar{u}, \bar{v})$ is also a relatively strongest weight in Λ_c for $\mathrm{DMU_o}$ and a strongest weight in Λ_c for $\mathrm{DMU_o}$.

4 | Illustrative Example

In this section, we use the data recorded in *Table 1* to illustrate how approaches introduced in Section 3 perform. These correspond to 13 DMUs, whose efficiency is assessed using one input and two outputs.

Table 1. Data set.

	Input	Output 1	Output 2
Unit 1	1	1	2.5
Unit 2	1	2.5	1
Unit 3	1	2.5	4
Unit 4	1	3.5	6
Unit 5	1	5	3.5
Unit.6	1	5.5	2.5
Unit.7	1	7	5.5
Unit.8	1	7	1
Unit.9	1	8	3.5
Unit.10	1	5.5	5.75
Unit.11	1	8	4.5
Unit.12	1	1.5	2
Unit.13	1	1.75	6

By using the data set from *Table 1*, we solve *Model (3)*, *Model (4)*, *Model (5)* and *Model (6)* for each DMU. The results are reported in *Table 2*. Every model diagnoses efficient units correctly. On the other hand, the efficiency scores for inefficient units will differ.

Table 2. Example results.

	Eff(2)	Eff (4)	Eff (5)	Eff (7)	Eff(9)
Unit 1	0.4166667	0.4166208	0.3846154	0.3076923	0.2500000
Unit 2	0.3125000	0.3124594	0.3846154	0.3076923	0.2500000
Unit 3	0.6699029	0.6699029	0.6153846	0.6153846	0.5833333
Unit 4	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit 5	0.6800000	0.6800000	0.6153846	0.6153846	0.5833333
Unit.6	0.6875000	0.6874406	0.6923077	0.6153846	0.5833333
Unit.7	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit.8	0.8750000	0.8747062	0.8461538	0.7692308	0.7500000
Unit.9	1.0000000	0.9999990	1.0000000	0.9230769	0.9166667
Unit.10	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit.11	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000
Unit.12	0.3398058	0.3398058	0.2307692	0.2307692	0.1666667
Unit.13	1.0000000	0.9999982	1.0000000	0.9230769	0.9166667

5 | Conclusion

In Section 3, we provided models to obtain nonnegative weights for inputs and outputs for the weights, the number of which DMUs for each one its virtual output does not exceed (is less than, if any) its virtual input be maximum, provided that for DMU under evaluation, the virtual output will be equal to the virtual input and the virtual input will be positive. We called these weights the relatively best weight (the relatively strongest weight, if any) for the DMU under evaluation, and if all the weights were positive, we called them the best weight (the strongest weight, if any) for the DMU under evaluation. The relatively best weight (the relatively strongest weight, if any) indicates the normal vector of a surface in the PPS with returns to scale constant assumption that the DMU under evaluation is on the surface and the maximum number of which DMUs their performance is no worse than (is better than) the DMU under evaluation separate from the rest of DMUs. Also in this paper, we presented the relationship between these definitions of efficiency with efficiency in the DEA models with constant returns to scale assumption. The normal vectors can be applied as a criterion for efficiency analysis and ranking of a set of peer DMUs with interval scale data. Especially the relatively strongest weight, both indicate extreme CCR-efficiency and provide a performance measure $\mathrm{DMU_o}$ with interval scale inputs and/or outputs. Also the strongest weight can be applied for ranking extreme CCR-efficient DMUs and CCR-inefficient DMUs.

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Data Availability

Data are available from the corresponding author upon reasonable request.

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