Research Annals of Industrial and Systems Engineering

www.raise.reapress.com

Res. Ann. Ind. Syst. Eng. Vol. 1, No. 3 (2024) 171–181.

Paper Type: Original Article

# A New Approach for Fixed Cost Allocation in DEA Based on the Value Efficiency Analysis

**Javad Gerami***

Department of Mathematics, Shiraz branch, Islamic Azad University, Shiraz, Iran; geramijavad@gmail.com.

**Citation:**

| | |
|---|---|
| *Received: 10 July 2024*<br>*Revised: 17 September 2024*<br>*Accepted: 14 November 2024* | Gerami, J. (2024). A new approach for fixed cost allocation in DEA Based on the value efficiency analysis. *Research annals of industrial and systems engineering, 1*(3), 171-181. |

## Abstract

Fixed cost allocation among Decision-Making Units (DMUs) should be based on a fair plan. This paper presents a new approach based on value efficiency analysis. In this regard, we first calculate the value efficiency scores of the DMUs by selecting the Most Preferred Solution (MPS) units. These units can be the units that have the best performance from the Decision-Maker (DM) point of view. In the following, we present an algorithm for providing a fixed cost allocation plan among DMUs based on the value efficiency analysis in Data Envelopment Analysis (DEA). Fixed cost allocation is done by choosing the efficiency invariance strategy. Value efficiency analysis was used to design a fixed cost allocation plan using the DMs preferred information.

**Keywords:** Data envelopment analysis, Fixed cost allocation, Value efficiency, Efficiency invariance.

## 1 | Introduction

The different DMUs can operate together. These units can be service organizations such as banks or manufacturing plants. Senior managers may want to divide a fixed cost among DMUs in many situations. This fixed cost allocation plan can be based on different strategies. One of these strategies is based on the efficiency invariance principle. This strategy states that the efficiency of DMUs will not change before and after the fixed cost allocation process. We now review recent studies on fixed cost allocation using Data Envelopment Analysis (DEA). Fixed cost allocation is one of the important issues in many organizations, including banks, commercial enterprises, and industrial firms. It helps managers take a fair perspective on the organization they manage and prevent the overall waste of resources. Additionally, fixed costs play a key role in decision-making processes related to pricing and determining profitability across different industries. Proper allocation of these costs among various Decision-Making Units (DMUs) is essential, such as allocating advertising costs among retailers and distributing health resources and equipment upgrades.

Corresponding Author: geramijavad@gmail.com

https://doi.org/10.22105/raise.v1i3.58

In this area, a group of studies has been developed based on the fundamental work of Cook and Kress [1], which introduce two main assumptions: “efficiency invariance” and “pareto-minimal.” The efficiency invariance principle states that after allocating fixed costs among DMUs, the efficiency of these units should not change compared to the situation before the allocation. A fixed or common cost is imposed on all DMUs in many DEA applications. The goal is to allocate these costs among different DMUs so each DMU bears its share. In Cook and Kress’s [1] proposed approach, several computational issues arise after fairly allocating shared costs that require solving linear programming problems. However, Jahanshahloo et al. [2] provided a simple model that resolves these problems easily without solving linear programming problems, thus proposing a straightforward method for cost allocation without computational complexity. This method can be applied to constant and variable returns to scale (CRS and VRS) technologies. It should be noted that in both Cook and Kress’s methods [1] and the method proposed in this article, the principle of “pareto-minimal” is not preserved. Subsequently, Cook and Zhu [3] extended the work of Cook and Kress [1] for various models. Lin [4], [5] also suggested modifications based on the approach of Cook and Zhu [3], although the changes were not very significant. Nevertheless, these studies emphasize the importance of fixed cost allocation and address concerns in this process, presenting algorithms and theorems that help ensure the correct allocation of costs.

Beasley [6] introduced an approach called efficiency maximization. This approach aims to improve the average efficiency of DMUs. This approach is beneficial in environments where balanced and simultaneous allocation among various DMUs is necessary. Beasley's approach, by optimizing efficiency, can aid in the optimal allocation of resources and prevent unfair allocation. Si et al. [7] developed methods for cost allocation using common weight and proportional sharing. This approach can be applied in systems seeking equitable cost allocation among DMUs. This model helps ensure that cost allocation minimizes cost gaps between units and maintains fairness in allocation. Mostafaei [8], in his research, particularly in the field of fixed cost allocation in practical applications like public services and production processes, addresses the problem of allocating shared costs using DEA. He introduced a new method for allocating fixed costs to DMUs using DEA. Due et al. [9] introduced an approach based on cross-efficiency. In this approach, cost allocation is conducted in such a way that the efficiency consideration of each unit directly impacts the efficiency of other units. Lin and Chen [10] used DEA to address resource and fixed cost allocation issues.

The researchers proposed a new sharing model in which fixed resources and targets are divided among DMUs. This model also includes two corresponding algorithms that can generate a unique allocation for each DMU. Furthermore, the proposed method can be extended to CRS and VRS technologies. Jahanshahloo et al. [11] proposed two important approaches for solving the fixed cost allocation problem in line with efficiency invariance and weight set principles. These methods are instrumental in fixed cost allocation in DEA applications. Li et al. [12] also addressed target setting and resource allocation issues, considering the principles of efficiency invariance and common weights in allocating resources and targets between DMUs. They proposed a new mechanism incorporating these principles in allocating multiple resources, particularly in situations involving multiple targets across units. Zhu et al. [13] proposed DEA models for measuring fixed cost performance in two-stage systems. This approach carefully considers input and output factors to allocate the best cost. This approach is suitable for complex systems with multi-stage processes where resources must be meticulously allocated. Li et al. [14] extended the traditional fixed cost allocation issue to systems with a two-stage network structure. Using DEA, researchers assess the relative efficiency of DMUs and make efficient cost allocation possible under a set of common weights. This approach allows DMUs to maximize efficiency by choosing different allocations and relative weights. Due [9] to the diverse allocations existing in the efficient set, allocation programs are optimized with a focus on operational unit sizes, and a minimax model and an algorithm are provided to reduce deviations between efficient allocation and size. An et al. [15] proposed an efficiency-based approach to solve fixed cost allocation issues in two-stage systems, extending this model to general systems. This approach is particularly important in scenarios where decision-making systems involve multiple stages or levels of operation. Utilizing this model in cooperative and non-cooperative systems can help optimize the allocation of costs while also maintaining system efficiency. Chu. et al. [16]

expanded the FCA method by incorporating the "full efficiency" principle, offering new fixed-cost allocation approaches in two-stage structures. They propose a range of possible allocations and models to consider competition between two-stage DMUs under a centralized framework. Leader-follower models and the concept of union satisfaction degree have also been utilized to provide stable and acceptable allocations. Dai et al. [17] introduced a two-stage incentive approach for allocating shared revenues or fixed costs.

The proposed method suggests cross-efficiency DEA for evaluating DMUs and incentive allocation, providing nonlinear allocation models and simple equations for global optimization. Practical aspects of these methods include improved performance and efficiency in information asymmetry decision-making environments. This approach motivates sub-units to enhance their performance. An et al. [18] identified shortcomings in Dai et al.'s [17] approach and proposed two alternative incentive mechanisms for allocating shared revenues or fixed costs. These mechanisms are designed under informational symmetry and asymmetry conditions and establish incentive productivity criteria. These mechanisms are tested on real data from a Chinese company based on the efficiency ratio. Chu et al. [19] emphasized fairness in allocation and presented a multi-objective model for cost allocation that considers the needs and preferences of DMUs. This model is particularly important when multiple stakeholders with different preferences are involved in decision-making. This approach can create a fair allocation that responds to the diverse needs and desires of DMUs. Yang et al. [20] examined discrepancies in cost allocation and sought to minimize the deviation between individual efficiency and total preferences. This approach can improve coordination among DMUs, make cost allocation mathematically optimal, and reduce disagreements. Zhang et al. [21] introduced aggressive game strategies for cost allocation in a decentralized environment. This approach is particularly useful in systems where DMUs operate independently. Utilizing game theory, this approach seeks agreements among DMUs to allocate costs to ensure fairness and efficiency.

Additionally, this approach demonstrates that after cost allocation, the average efficiencies converge towards the cross-efficiency of the aggressive game. Yang et al. [20] introduce a new DEA-based fixed cost allocation method that balances individual efficiency assurance goals and collective priority objectives simultaneously. This approach involves constructing a Priority Value Loss (PVL) index, which accurately measures the effects of priority considerations. Moreover, our generalized fixed cost allocation strategy minimizes PVL and provides a prioritized evaluation process for selecting the final allocation plan.

Value efficiency analysis is needed to apply the DMs (senior managers of companies) opinion in the DEA. Halmé et al. [22] created a DEA-based operational method that incorporates DM's preference information to identify the most preferred input-output configuration. In this study, they develop a necessary procedure and theory for integrating preference information in a novel way within the context of DMU efficiency analysis. The efficiency of DMUs is defined based on the essence of DEA, which is complemented with preference information from the DM regarding the desired structure of inputs and outputs. The proposed method starts by helping the Decision-Maker (DM) identify the most preferred combination of input and output among efficient DMUs in DEA. Then, assuming that the Most Preferred Solution (MPS) maximizes the DM's underlying objective function (unknown), the indifference curve of the function at that point is approximated by its potential tangent hyperplanes. Finally, efficiency scores for each DMU are calculated, and inefficient units are compared with those with a similar value to the preferred ones. The resulting efficiency scores are optimistic approximations of the real scores. This method and the resulting efficiency scores immediately apply to solving practical problems. In value efficiency analysis, the efficiency of units is calculated based on a new efficiency frontier. In this way, the DM selects several units as the units with the highest efficiency (MPS units). If the MPS units are observed, we call them the Most Perfect Units (MPUs). These Units have the best performance among the DMUs. For example, in evaluating a bank branch complex, some banks have the best performance from a management point of view, and based on value efficiency analysis, the efficiency of other branches is obtained based on the efficiency of these units. In the process of value efficiency analysis, the value efficiency frontier is introduced instead of the efficiency frontier, and an efficiency value called value efficiency corresponding to each unit is obtained. Similarly, we can perform value efficiency analysis based on economic efficiency instead of technical efficiency [23]. Halmé. et al [22] proposed a non-convex value efficiency analysis and its application to bank branch. Gerami [24] proposed an interactive procedure for improving the estimate of value efficiency in DEA. Gerami et al. [25] propose a novel geometric interpretation for value efficiency when plugging it into radial and non-radial DEA models under VRS technology. Gerami et al. [26] proposed a generalized inverse DEA model based on value efficiency for firm restructuring.

This paper mainly introduces the MPS units and presents value efficiency evaluation models in VRS technology. We also present a new fixed-cost allocation approach based on the efficiency invariance strategy and value efficiency analysis. We illustrate our models by using a case study to evaluate commercial banks. Based on the presented approach, we can obtain a fair allocation for banks.

The rest of the paper is organized as follows: in the second section, we present the basics of fixed cost allocation with the efficiency invariance strategy of DMUs and value efficiency analysis. The third section presents an algorithm for providing a fixed cost allocation plan based on value efficiency. In the fifth section, we use the presented algorithm to allocate a fixed cost among DMUs that operate under a single management. In the fifth section, we present the results of the presented models.

## 2 | Background

Let n DMU as $\mathrm{DMU_j}$, $j = 1, \dots, n$. These DMUs use of vectors $X_j = (x_{1j}, \dots, x_{mj}) \in R^m_+$ for producting vectors $Y_j = (y_{1j}, \dots, y_{sj}) \in R^s_+$. In this section, we briefly describe Cook and Zhu's method [3] based on the efficiency invariance principle for the issue of fixed cost allocation. Assume that the fixed does not change. According to this method, the efficiency score of all DMUs does not change before and after the fixed cost allocation process. We consider the unit under evaluation to be $\mathrm{DMU_o} = (X_o, Y_o)$. We proposed the BCC model under VRS technology and output-oriented as follows.

$$\begin{aligned}
&\min \textstyle\sum_{i=1}^{m} v_i x_{io} + v_o,\\
&\text{s.t. } \textstyle\sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} + v_o \ge 0, \quad j = 1, \dots, n,\\
&\quad \textstyle\sum_{r=1}^{s} u_r y_{ro} = 1,\\
&\quad v_i \ge 0, i = 1, \dots, m,\\
&\quad u_r \ge 0, r = 1, \dots, s, \quad v_o \text{ is free in sign.}
\end{aligned} \tag{1}$$

In *Model (1)*, $v_i$, $i = 1, \dots, m$, $u_r$, $r = 1, \dots, s$ are the weights of input and output components, respectively the *Model's Dual (1)* is as follows.

$$\begin{aligned}
&\varphi_o^* = \max\ \varphi_o,\\
&\text{s.t. } \textstyle\sum_{j=1}^{n} \lambda_{jo} x_{ij} \le x_{io}, \quad i = 1, \dots, m,\\
&\textstyle\sum_{j=1}^{n} \lambda_{jo} y_{rj} \ge \varphi_o y_{ro}, \quad r = 1, \dots, s,\\
&\textstyle\sum_{j=1}^{n} \lambda_{jo} = 1,\\
&\lambda_{jo} \ge 0, \quad j = 1, \dots, n.
\end{aligned} \tag{2}$$

In *Model (2)*, $\lambda_{jo}$ shows the intensity variable. he $\varphi_o$ is xxpansion variable of outputs.

**Definition 1.** $\mathrm{DMU_0}$ is efficient under VRS if $\varphi_o^* = 1$. Otherwise, it is inefficient.

In the allocation plan, we want to allocate the fixed cost allocation as R amount among the $\mathrm{DMU_j}$, $j = 1, \dots, n$. uppose the cost allocated to $\mathrm{DMU_j}$, $j = 1, \dots, n$ is as $R_j$, $j = 1, \dots, n$. e put $\sum_{j=1}^{n} R_j = R$. e Consider the cost allocated to each DMU as a new input. In this case, *Model (1)* in the evaluation of $\mathrm{DMU_0}$ is as follows:

$$\begin{aligned}
&\min \textstyle\sum_{i=1}^{m} v_i x_{io} + v_o + v_{m+1} R_o,\\
&\text{s.t. } \textstyle\sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} + v_o + v_{m+1} R_j \ge 0, \quad j = 1, \dots, n,
\end{aligned} \tag{3}$$

$\sum_{r=1}^{s} u_r y_{ro} = 1,$

$\sum_{j=1}^{n} R_j = R,$

$v_i \geq 0, i = 1, \dots, m, \quad R_j, j = 1, \dots, n,$

$u_r \geq 0, r = 1, \dots, s, \quad v_o$ is free in sign.

We consider $v_{m+1}$ as the weight of the new input, namely $R_j, \ j = 1, \dots, n$. These variables are constant. When *Model (3)* is a linear programming, we will have a feasible non-allocation plan by considering $v_{m+1} = 0$. e put $v_{m+1} > 0$. The dual corresponding to *Model (3)* is as follows.

$$\begin{aligned}
&\varphi_o^{CRA^*} = \max\ \varphi_o^{CRA},\\
&\text{s.t. } \sum_{j=1}^{n} \lambda_{jo} x_{ij} \leq x_{io}, \quad i = 1, \dots, m,\\
&\sum_{j=1}^{n} \lambda_{jo} R_j \leq R_o,\\
&\sum_{j=1}^{n} \lambda_{jo} y_{rj} \geq \varphi_o y_{ro}, \quad r = 1, \dots, s,\\
&\sum_{j=1}^{n} \lambda_{jo} = 1,\\
&\lambda_{jo} \geq 0, R_j \geq o, \ j = 1, \dots, n.
\end{aligned} \tag{4}$$

Cook and Kress [1] presented the efficiency invariance principle. his principle states that the value of the *Model's Objective Function (1)* is equal to the value of the *Model's Objective Function (3)*, or realizing the efficiency invariance principle when we solve the *Model (3)* using the simplex method the variable $v_{m+1}$ must be out of the basis [2].

**Definition 2.** Metavariable $v_{m+1}$ remains out of the basis n; namely t, the reduced cost is nonnegative. Then we will have

$$c_{v_{m+1}} - z_{v_{m+1}} = c_{v_{m+1}} - c_B B^{-1} A \geq 0$$

Then

$$R_o - \sum_{j=1}^{n} \lambda_{jo}^* R_j \geq 0 \text{ or } \sum_{j=1}^{n} \lambda_{jo}^* R_j \leq R_o. \tag{5}$$

The variables $\lambda_{jo}^*, j = 1, \dots, n$, are the optimal dual variables of the *Model (3)* [1]. For the values of the *Objective Functions of Models (1)* and *(3)* to be equal, the second constraint in *Model (4)* must be redundant, meaning that $\lambda_{jo}^*, j = 1, \dots, n$ in *Relation (5)* must be the optimal solution of *Model (2)*. Then, $R_j, \ j = 1, \dots, n$ should satisfy in the $\sum_{j=1}^{n} \lambda_{jo}^* R_j \leq R_o$. This allocation is not unique. If the fixed cost is distributed among the inefficient DMUs completely in any proportion, then efficiency scores do not change, and the assumption of invariance is established [1]. Then, they defined another condition under the title of the Input Pareto-Minimality, which is as follows:

**Definition 3.** The input Pareto-minimality in fixed-cost allocation means that no cost can be transferred from one DMU to another without violating the invariance.

To meet efficiency invariance and the input pareto-minimality principle, the constraint $\sum_{j=1}^{n} \lambda_{jo}^* R_j = R_o$ must be in place for all inefficient DMUs. Suppose $(\varphi_o^*, \lambda_{jo}^*, j = 1, \dots, n)$ is an optimal solution of *Model (2)*, then we define $M = \{j | \lambda_{jo}^* > 0 \text{ in the optimal solution of model (2)}\}$. This set is used as a reference set for $DMU_o$.

The other dual variables are equal to zero according to complementary slackness. Lin and Chen [10] showed that this economic interpretation of pareto-minimality for the equality constraints is unsuitable, and they considered it a practical feasibility assumption. his equation makes exceptions for the possible inefficiency (non-zero slack) from the cost allocation plane. Then we put $\sum_{j \in M} \lambda_{jo}^* R_j = R_o$. his constraint ensures that the cost allocation is not completely distributed among inefficient DMUs. Put $\mathrm{NE}$ shows the set includes the index corresponding to the inefficient DMUs. Suppose we have an appropriate cost allocation plan of the form $R_j,\ j = 1, \dots, n$ that $\sum_{j\in M} \lambda_{jo}^* R_j = R_o$ and $\lambda_{jo}^* > 0, j \in M$. Also, we must have $\sum_{j=1}^{n} R_j = R$. Considering the above, we seek to present a fair allocation plan based on value efficiency analysis.

## 3 | Value Efficiency Analysis and Fixed Cost Allocation

In value efficiency analysis, the efficiency of DMUs is obtained based on a set of units called MPUs. These units are the units that, in the opinion of the manager, have the best performance. For this purpose, we evaluate $\mathrm{DMU_o}$ based on *Model (2)*. Suppose $(\varphi_o^*, \lambda_{jo}^*, j = 1, \dots, n)$ is an optimal solution of *Model (2)*. Almé et al. [22] proposed a *Model (5)* to calculate the value efficiency of $\mathrm{DMU_o}$ as follows.

$$
\begin{aligned}
&\varphi_o^{va*} = \max\ \varphi_o^{va},\\
&\text{s.t. } \textstyle\sum_{j=1}^{n} \mu_{jo} x_{ij} \le x_{io}, \quad i = 1, \dots, m,\\
&\textstyle\sum_{j=1}^{n} \mu_{jo} y_{rj} - \varphi_o^{va} y_{ro} \ge y_{ro}, \quad r = 1, \dots, s,\\
&\textstyle\sum_{j=1}^{n} \mu_{jo} = 1,\\
&\mu_{jo}, \text{ is free in sign if } \lambda_{jo}^* > 0,\\
&\mu_{jo} \ge 0, \text{ if } \lambda_{jo}^* = 0.
\end{aligned}
\tag{5}
$$

**Definition 4.** $\mathrm{DMU_0}$ is value efficient under VRS if $\varphi_o^{va*} = 0$. Otherwise it is inefficient.

The value efficiency score of $\mathrm{DMU_0}$ is defined as $\frac{1}{1+\varphi_o^{va*}}$.

Suppose we show MPU units as $\mathrm{MP}$. The set can be $\mathrm{MP} \subseteq \mathrm{M}$ or selected from the set of observed DMUs based on the manager's opinion. The *Model (5)* is converted as follows.

$$
\begin{aligned}
&\varphi_o^{va*} = \max\ \varphi_o^{va},\\
&\text{s.t. } \textstyle\sum_{j=1}^{n} \mu_{jo} x_{ij} \le x_{io}, \quad i = 1, \dots, m,\\
&\textstyle\sum_{j=1}^{n} \mu_{jo} y_{rj} - \varphi_o^{va} y_{ro} \ge y_{ro}, \quad r = 1, \dots, s,\\
&\textstyle\sum_{j=1}^{n} \mu_{jo} = 1,\\
&\mu_{jo}, \text{ is free in sign if } j \in \mathrm{MP},\\
&\mu_{jo} \ge 0, \text{ if } j \notin \mathrm{MP}.
\end{aligned}
\tag{6}
$$

We first solve *Model (2)* for each of the DMUs. Suppose that the set of inefficient units is represented as $\mathrm{NE}$. Suppose $(\varphi_q^*, \lambda_{jq}^*, q \in \mathrm{NE})$ is an optimal solution of *Model (2)* for $\mathrm{DMU_q}, q \in \mathrm{NE}$, e construct the reference set corresponding to the inefficient units ($\mathrm{DMU_q}, q \in \mathrm{NE}$) as follows:

$$
M_q = \{ j | \lambda_{jq}^* > 0 \text{ in the optimal solution of model (2) for } \mathrm{DMU_q}, q \in \mathrm{NE} \}.
$$

We can develop an algorithm for a fixed cost allocation plane based on the value efficiency analysis. For this purpose, we assume that $(\hat{\varphi}_q, \hat{\mu}_{jq}, q \in \mathrm{NE})$ is an optimal solution of *Model (6)*. so we put $\bar{R} = \frac{R}{n}$, e put $\mathrm{MP_q} \subseteq M_q, q \in \mathrm{NE}$. To find the optimal fixed cost allocation, we propose the following model.

$$
\begin{aligned}
&\min \textstyle\sum_{j=1}^{n} \left| R_j - \bar{R} \right|,\\
&\text{s.t. } \textstyle\sum_{j \in \mathrm{MP_q}} \hat{\mu}_{jq} R_j = R_q, \ q \in \mathrm{NE},
\end{aligned}
\tag{7}
$$

$\sum_{j=1}^{n} R_j = R,$

$R_j \geq o,\ j = 1, \dots, n.$

*Model (7)* is nonlinear because of its objective function. e can replace it with a linear objective function. Or example $\min W$. The proposed algorithm is as follows:

**Step 1.** Initially, we form a set $MP_q$ for $q \in NE$.

**Step 2.** Solve *Model (6)* and obtain $(\hat{\varphi}_q, \hat{\mu}_{jq}, q \in NE)$ as an optimal solution

**Step 3.** Solve *Model (7)* and obtain the fair fixed cost allocation plane.

A fixed cost allocation plan has a unique solution if the following linear equation system has a unique solution.

$\sum_{j \in MP_q} \hat{\mu}_{jq} R_j = R_q,\ q \in NE,$

$\sum_{j=1}^{n} R_j = R.$

A bove linear equation system has $|NE| + 1$ equations and $n$ variables.

# 4 | A Numerical Example

We now present a numerical example previously used by Cook and Kress [1] to demonstrate the fixed cost allocation plan presented in this paper. *Table 2* illustrates 12 DMUs that each have three inputs and two outputs. First, we solve *Model (2)* for each DMU to form the set NE.

**Table 2. The data set is in the numerical example.**

| DMUs | Input 1 | Input 2 | Input 3 | Output 1 | Output 2 | The Efficiency Scores of the Model (2) | Units in the Reference Set |
|---|---|---|---|---|---|---|---|
| 1 | 350 | 39 | 9 | 67 | 751 | 0.7825 | DMU6, DMU8, DMU9, DMU12 |
| 2 | 298 | 26 | 8 | 73 | 611 | 0.9453 | DMU4, DMU5, DMU6, DMU8 |
| 3 | 422 | 31 | 7 | 75 | 584 | 0.8921 | DMU5, DMU6, DMU12 |
| 4 | 281 | 16 | 9 | 70 | 665 | 1 | DMU4 |
| 5 | 301 | 16 | 6 | 75 | 445 | 1 | DMU5 |
| 6 | 360 | 29 | 17 | 83 | 1070 | 1 | DMU6 |
| 7 | 540 | 18 | 10 | 72 | 457 | 0.9445 | DMU5, DMU6 |
| 8 | 276 | 33 | 5 | 78 | 590 | 1 | DMU8 |
| 9 | 323 | 25 | 5 | 75 | 1074 | 1 | DMU9 |
| 10 | 444 | 64 | 6 | 74 | 1072 | 0.8941 | DMU12 |
| 11 | 323 | 25 | 5 | 25 | 350 | 0.3333 | DMU9 |
| 12 | 444 | 64 | 6 | 104 | 1199 | 1 | DMU12 |

According to column 7 in *Table 2*, DMUs 4, 5, 6, 8, 9, and 12 are efficient, and DMUs 1, 2, 3, 7, 10, and 11 are inefficient. Column 8 of *Table 2* shows the set of DMUs in the reference set corresponding to each DMU.

We assume that we have a cost fixed equal to 100 for allocation; to obtain the optimal allocation plan, we solve the *Model (6)* for inefficient DMUs. For this purpose, we form a set MP for inefficient DMUs, namely MPUs units, e set

$MP_1 = \{DMU6, DMU8, DMU9, DMU12\}, MP_2 = \{DMU4, DMU5, DMU6, DMU8\},$

$MP_3 = \{DMU5, DMU6, DMU12\},\ MP_7 = \{DMU5, DMU6\}, MP_{10} = \{DMU12\},\ MP_{11} = \{DMU9\}.$

*Table 3* shows the results of *Model (6)* for inefficient units. e obtain a cost allocation by the last column of *Table 3*. It can be easily seen that this cost allocation satisfies invariance and pareto-minimalitytions in Cook and Kress [1]. The valve valuesiency scores of DMUs are the same as before and after allocating fixed costs to DMUs according to *Tables 2* and *3*. *Table 3* shows the results of *Model (6)* for inefficient units. e obtain a cost allocation by the last column of *Table 3*. This cost allocation can easily satisfy the invariance and pareto-minimalityonditions in Cook and Kress [1]. The efficiency scores of DMUs are measured before and after allocating fixed costs to DMUs according to *Tables 2* and *3*.

Now, for sensitivity analysis, we change the MPUs units corresponding to each of the inefficient units as follows:

$$\mathrm{MP}_1 = \{\mathrm{DMU6}, \mathrm{DMU8}, \mathrm{DMU9}, \mathrm{DMU12}\}, \mathrm{MP}_2 = \{\mathrm{DMU4}, \mathrm{DMU5}, \mathrm{DMU6}, \mathrm{DMU8}\},$$

$$\mathrm{MP}_3 = \{\mathrm{DMU4}, \mathrm{DMU5}, \mathrm{DMU6}, \mathrm{DMU8}\}, \mathrm{MP}_7 = \{\mathrm{DMU4}, \mathrm{DMU5}, \mathrm{DMU6}, \mathrm{DMU8}\},$$

$$\mathrm{MP}_{10} = \{\mathrm{DMU6}, \mathrm{DMU8}, \mathrm{DMU9}, \mathrm{DMU12}\}, \mathrm{MP}_{11} = \{\mathrm{DMU6}, \mathrm{DMU8}, \mathrm{DMU9}, \mathrm{DMU12}\}.$$

In this case, the cost allocated to the units is as follows.

$$\mathrm{R}_1^* = 0,\ \mathrm{R}_2^* = 3.0684,\ \mathrm{R}_3^* = 27.9827,\ \mathrm{R}_4^* = 0,\ \mathrm{R}_5^* = 11.8561,\ \mathrm{R}_6^* = 0,\ \mathrm{R}_7^* = 56.691,$$
$$\mathrm{R}_8^* = 0, \mathrm{R}_9^* = 0, \mathrm{R}_{10}^* = 0, \mathrm{R}_{11}^* = 0.4019, \mathrm{R}_{12}^* = 0.$$

And the value efficiency scors are as

$$\varphi_1^{\mathrm{va}*} = 0.7825,\ \varphi_2^{\mathrm{va}*} = 0.9453,\ \varphi_3^{\mathrm{va}*} = 0.7867,\quad \varphi_4^{\mathrm{va}*} = 1,\ \varphi_5^{\mathrm{va}*} = 1,\ \varphi_6^{\mathrm{va}*} = 1,$$
$$\varphi_7^{\mathrm{va}*} = 0.678, \varphi_8^{\mathrm{va}*} = 1, \varphi_9^{\mathrm{va}*} = 1, \varphi_{10}^{\mathrm{va}*} = 0.7557, \varphi_{11}^{\mathrm{va}*} = 0.3324, \varphi_{12}^{\mathrm{va}*} = 1.$$

**Table 3. The results of Model (6).**

| DMUs | The Value Efficiency Scores of the Model (6) | Non-Zero Optimal Solution | Fixed Cost |
|---|---|---|---|
| 1 | 0.7825 | $\mu_6^* = 0.2219, \mu_8^* = 0.3048, \mu_9^* = 0.1996, \mu_{12}^* = 0.2737$ | 10.6847 |
| 2 | 0.9453 | $\mu_5^* = 0.111, \mu_5^* = 0.2588, \mu_6^* = 0.1783, \mu_8^* = 0.4519$ | 0 |
| 3 | 0.8921 | $\mu_5^* = 0.6212, \mu_6^* = 0.0909, \mu_{12}^* = 0.2879$ | 11.2391 |
| 4 | 1 | $\mu_4^* = 1$ | 0 |
| 5 | 1 | $\mu_5^* = 1$ | 0 |
| 6 | 1 | $\mu_6^* = 1$ | 0 |
| 7 | 0.9445 | $\mu_5^* = 0.8462, \mu_6^* = 0.1538$ | 0 |
| 8 | 1 | $\mu_8^* = 1$ | 0 |
| 9 | 1 | $\mu_9^* = 1$ | 0 |
| 10 | 0.8941 | $\mu_{12}^* = 1$ | 39.0381 |
| 11 | 0.3333 | $\mu_9^* = 1$ | 0 |
| 12 | 1 | $\mu_{12}^* = 1$ | 39.0381 |

As can be seen, the allocated cost is allocated to both efficient and inefficient DMUs.

Now, we propose the results of fixed cost allocation based on the input-oriented model. According to column 7 in *Table 4*, DMUs 4, 5, 6, 8, 9, 11, and 12 are efficient, and DMUs 1, 2, 3, 7, and 10 are inefficient. Column 8 of *Table 2* shows the set of DMUs in the reference set corresponding to each of the DMUs of the input-oriented model under VRS technology.

**Table 4. The data set is in the numerical example.**

| DMUs | Input 1 | Input 2 | Input 3 | Output 1 | Output 2 | The Efficiency Scores BCC Model in the Input-Oriented | Units in the Reference Set |
|---|---|---|---|---|---|---|---|
| 1 | 350 | 39 | 9 | 67 | 751 | 0.8292 | DMU4, DMU8, DMU9 |
| 2 | 298 | 26 | 8 | 73 | 611 | 0.9348 | DMU4, DMU8 |
| 3 | 422 | 31 | 7 | 75 | 584 | 0.7483 | DMU5, DMU8, DMU9 |
| 4 | 281 | 16 | 9 | 70 | 665 | 1 | DMU4 |
| 5 | 301 | 16 | 6 | 75 | 445 | 1 | DMU5 |
| 6 | 360 | 29 | 17 | 83 | 1070 | 1 | DMU6 |
| 7 | 540 | 18 | 10 | 72 | 457 | 0.8889 | DMU4, DMU5 |
| 8 | 276 | 33 | 5 | 78 | 590 | 1 | DMU8 |
| 9 | 323 | 25 | 5 | 75 | 1074 | 1 | DMU9 |
| 10 | 444 | 64 | 6 | 74 | 1072 | 0.8333 | DMU9 |
| 11 | 323 | 25 | 5 | 25 | 350 | 1 | DMU9 |
| 12 | 444 | 64 | 6 | 104 | 1199 | 1 | DMU12 |

As we said before, we assume that we have a cost fixed equal to 100 for allocation; to obtain the optimal allocation plan, we solve the BCC model in the input-oriented for inefficient DMUs. For this purpose, we form set MP for inefficient DMUs, namely MPUs units, e set

$\mathrm{MP}_1 = \{\mathrm{DMU4}, \mathrm{DMU8}, \mathrm{DMU9}\}$,$\mathrm{MP}_2 = \{\mathrm{DMU4}, \mathrm{DMU8}\}$,$\mathrm{MP}_3 = \{\mathrm{DMU5}, \mathrm{DMU8}, \mathrm{DMU9}\}$,

$\mathrm{MP}_7 = \{\mathrm{DMU4}, \mathrm{DMU5}\}$, $\mathrm{MP}_{10} = \{\mathrm{DMU9}\}$, $\mathrm{MP}_{11} = \{\mathrm{DMU9}\}$.

*Table 5* shows the value efficiency of the BCC DEA model, which results in the input orientation for inefficient units. e obtain a cost allocation by the last column of *Table 5*. It can be easily seen that this cost allocation satisfies the invariance and pareto-minimality conditions in Cook and Kress [1]. The values efficiency scores of DMUs are the same before and after allocating fixed costs to DMUs according to *Tables 4 and 5*.

**Table 5. The results of value efficiency BCC DEA model in the input-oriented.**

| DMUs | The Value Efficiency Scores of the Value Efficiency DEA Model in the Input-Oriented | Non-Zero Optimal Solution | Fixed Cost |
|---|---|---|---|
| 1 | 0.8292 | $\mu_4^* = 0.6158$, $\mu_8^* = 0.147$, $\mu_9^* = 0.2372$ | 6.8271 |
| 2 | 0.9348 | $\mu_4^* = 0.5115$, $\mu_8^* = 0.4885$, | 9.2453 |
| 3 | 0.7483 | $\mu_5^* = 0.238$, $\mu_8^* = 0.0423$, $\mu_9^* = 0.7172$ | 8.6547 |
| 4 | 1 | $\mu_4^* = 1$ | 9.3878 |
| 5 | 1 | $\mu_5^* = 1$ | 10.4875 |
| 6 | 1 | $\mu_6^* = 1$ | 10.8762 |
| 7 | 0.8889 | $\mu_4^* = 0.6$, $\mu_5^* = 0.4$ | 9.6921 |
| 8 | 1 | $\mu_8^* = 1$ | 9.5998 |
| 9 | 1 | $\mu_9^* = 1$ | 9.3472 |
| 10 | 0.8333 | $\mu_9^* = 1$ | 5.3421 |
| 11 | 1 | $\mu_9^* = 1$ | 0.5428 |
| 12 | 1 | $\mu_{12}^* = 1$ | 9.9974 |

# 5 | Conclusions

This paper proposes a DEA approach to cost allocation problems based on the value efficiency analysis. We can incorporate the DMs opinion into the fixed cost allocation process using value efficiency analysis. We proposed an algorithm for a fixed cost allocation plane using VRS technology. e also presents a new approach for fixed cost allocation based on the efficiency invariance strategy and value efficiency analysis. One of the strengths of the presented approach is that based on the DM's preferred information, we can allocate the fixed cost among all DMUs, including efficient and inefficient DMUs. We illustrate our models by providing a numerical example. e can extend the presented algorithm to obtain a unique cost allocation scheme. e can also develop models for the two-stage network structure in DEA.

# Conflict of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

# Data Availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

# Funding

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

# References

[1] Cook, W. D., & Kress, M. (1999). Characterizing an equitable allocation of shared costs: A DEA approach. *European journal of operational research*, *119*(3), 652–661. https://doi.org/10.1016/S0377-2217(98)00337-3

[2] Jahanshahloo, G. R., Lotfi, F. H., Shoja, N., & Sanei, M. (2004). An alternative approach for equitable allocation of shared costs by using DEA. *Applied mathematics and computation*, *153*(1), 267–274. https://doi.org/10.1016/S0096-3003(03)00631-3

[3] Cook, W. D., & Zhu, J. (2005). Allocation of shared costs among decision making units: A DEA approach. *Computers & operations research*, *32*(8), 2171–2178. https://doi.org/10.1016/j.cor.2004.02.007

[4] Lin, R. (2011). Allocating fixed costs or resources and setting targets via data envelopment analysis. *Applied mathematics and computation*, *217*(13), 6349–6358. https://doi.org/10.1016/j.amc.2011.01.008

[5] Lin, R. (2011). Allocating fixed costs and common revenue via data envelopment analysis. *Applied mathematics and computation*, *218*(7), 3680–3688. https://doi.org/10.1016/j.amc.2011.09.011

[6] Beasley, J. E. (2003). Allocating fixed costs and resources via data envelopment analysis. *European journal of operational research*, *147*(1), 198–216. https://doi.org/10.1016/S0377-2217(02)00244-8

[7] Si, X., Liang, L., Jia, G., Yang, L., Wu, H., & Li, Y. (2013). Proportional sharing and DEA in allocating the fixed cost. *Applied mathematics and computation*, *219*(12), 6580–6590. https://doi.org/10.1016/j.amc.2012.12.085

[8] Mostafaee, A. (2013). An equitable method for allocating fixed costs by using data envelopment analysis. *Journal of the operational research society*, *64*(3), 326–335. https://doi.org/10.1057/jors.2012.56

[9] Du, J., Cook, W. D., Liang, L., & Zhu, J. (2014). Fixed cost and resource allocation based on DEA cross-efficiency. *European journal of operational research*, *235*(1), 206–214. https://doi.org/10.1016/j.ejor.2013.10.002

[10] Lin, R., & Chen, Z. (2016). Fixed input allocation methods based on super CCR efficiency invariance and practical feasibility. *Applied mathematical modelling*, *40*(9–10), 5377–5392. https://doi.org/10.1016/j.apm.2015.06.039

[11] Jahanshahloo, G. R., Sadeghi, J., & Khodabakhshi, M. (2017). Proposing a method for fixed cost allocation using DEA based on the efficiency invariance and common set of weights principles. *Mathematical methods of operations research*, *85*, 223–240. https://doi.org/10.1007/s00186-016-0563-z

[12] Li, F., Song, J., Dolgui, A., & Liang, L. (2017). Using common weights and efficiency invariance principles for resource allocation and target setting. *International journal of production research*, *55*(17), 4982–4997. https://doi.org/10.1080/00207543.2017.1287450

[13] Zhu, W., Zhang, Q., & Wang, H. (2019). Fixed costs and shared resources allocation in two-stage network DEA. *Annals of operations research*, *278*, 177–194. https://doi.org/10.1007/s10479-017-2599-8

[14] Li, Y., Li, F., Emrouznejad, A., Liang, L., & Xie, Q. (2019). Allocating the fixed cost: An approach based on data envelopment analysis and cooperative game. *Annals of operations research*, *274*, 373–394. https://doi.org/10.1007/s10479-018-2860-9

[15] An, Q., Wang, P., Emrouznejad, A., & Hu, J. (2020). Fixed cost allocation based on the principle of efficiency invariance in two-stage systems. *European journal of operational research*, *283*(2), 662–675. https://doi.org/10.1016/j.ejor.2019.11.031

[16] Chu, J., Wu, J., Chu, C., & Zhang, T. (2020). DEA-based fixed cost allocation in two-stage systems: leader-follower and satisfaction degree bargaining game approaches. *Omega*, *94*, 102054. https://doi.org/10.1016/j.omega.2019.03.012

[17] Dai, Q., Li, Y., Lei, X., & Wu, D. (2021). A DEA-based incentive approach for allocating common revenues or fixed costs. *European journal of operational research*, *292*(2), 675–686. https://doi.org/10.1016/j.ejor.2020.11.006

[18] An, Q., Tao, X., Xiong, B., & Chen, X. (2022). Frontier-based incentive mechanisms for allocating common revenues or fixed costs. *European journal of operational research*, *302*(1), 294–308. https://doi.org/10.1016/j.ejor.2021.12.039

[19] Chu, J., Su, W., Li, F., & Yuan, Z. (2023). Individual rationality and overall fairness in fixed cost allocation: An approach under DEA cross-efficiency evaluation mechanism. *Journal of the operational research society*, *74*(3), 992–1007. https://doi.org/10.1080/01605682.2022.2079434

[20] Yang, J., Li, D., & Li, Y. (2024). A generalized data envelopment analysis approach for fixed cost allocation with preference information. *Omega*, *122*, 102948. https://doi.org/10.1016/j.omega.2023.102948

[21] Zhang, D., Wu, H., Li, F., & Song, Y. (2024). Fixed cost allocation based on a data envelopment analysis aggressive game approach. *Computers & industrial engineering*, *193*, 110316. https://doi.org/10.1016/j.cie.2024.110316

[22] Halme, M., Korhonen, P., & Eskelinen, J. (2014). Non-convex value efficiency analysis and its application to bank branch sales evaluation. *Omega*, *48*, 10–18. https://doi.org/10.1016/j.omega.2014.04.002

[23] Joro, T., Korhonen, P., & Zionts, S. (2003). An interactive approach to improve estimates of value efficiency in data envelopment analysis. *European journal of operational research*, *149*(3), 688–699. https://doi.org/10.1016/S0377-2217(02)00458-7

[24] Gerami, J. (2019). An interactive procedure to improve estimate of value efficiency in DEA. *Expert systems with applications*, *137*, 29–45. https://doi.org/10.1016/j.eswa.2019.06.061

[25] Gerami, J., Mozaffari, M. R., Wanke, P. F., & Correa, H. L. (2022). Improving information reliability of non-radial value efficiency analysis: An additive slacks based measure approach. *European journal of operational research*, *298*(3), 967–978. https://doi.org/10.1016/j.ejor.2021.07.036

[26] Gerami, J., Mozaffari, M. R., Wanke, P. F., & Correa, H. L. (2023). A generalized inverse DEA model for firm restructuring based on value efficiency. *IMA journal of management mathematics*, *34*(3), 541–580. https://doi.org/10.1093/imaman/dpab043